



Computer
Science

CSC196: Analyzing Data

Mathematical Expectation

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Outline

- Mean of a Random Variable
- Variance & Covariance of a Random Variable
- Chebyshev's Theorem

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- Variance & Covariance of a Random Variable
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Measures of Location

Recall the sample mean is the arithmetic mean of the data.

Suppose that the observations in a sample are x_1, x_2, \dots, x_n . The **sample mean**, denoted by \bar{x} , is

$$\bar{x} = \sum_{i=1}^n \frac{x_i}{n} = \frac{x_1 + x_2 + \dots + x_n}{n}.$$

We often refer to this as the average of the data.

Example: Sample Mean

Suppose we toss 2 fair coins a total of 15 times and observe:

- 0 heads 4 times
- 1 head 7 times
- 2 heads 5 times

Frequency of each event

$$(0) \binom{4}{16} + (1) \binom{7}{16} + (2) \binom{5}{16} = 1.06.$$

Value of each event

Notice we can never see 1.06 heads

What happens if we do this 1,000 times? 100,000 times?

Moments of RVs: Expected Value

Definition The expected value of a RV X , denoted by $\mathbf{E}[X]$, is:

$$\mu_X = \mathbf{E}[X] = \sum_x x P(X = x)$$

if X is discrete. If X is continuous then the expected value is given by:

$$\mu_X = \mathbf{E}[X] = \int x p(X = x) dx$$

Summation / integral over all values in domain of X

Example Let X be the sum of two fair dice, then:

$$\mathbf{E}[X] = 2 \cdot \frac{1}{36} + 3 \cdot \frac{1}{18} + \dots + 12 \cdot \frac{1}{36} = 7$$

Example: Expected Value

Example 4.1: A lot containing 7 components is sampled by a quality inspector; the lot contains 4 good components and 3 defective components. A sample of 3 is taken by the inspector. Find the expected value of the number of good components in this sample.

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Solution: Let X represent the number of good components in the sample. The probability distribution of X is

$$f(x) = \frac{\binom{4}{x} \binom{3}{3-x}}{\binom{7}{3}}, \quad x = 0, 1, 2, 3.$$

Simple calculations yield $f(0) = 1/35$, $f(1) = 12/35$, $f(2) = 18/35$, and $f(3) = 4/35$. Therefore,

$$\mu = E(X) = (0) \left(\frac{1}{35} \right) + (1) \left(\frac{12}{35} \right) + (2) \left(\frac{18}{35} \right) + (3) \left(\frac{4}{35} \right) = \frac{12}{7} = 1.7.$$

Thus, if a sample of size 3 is selected at random over and over again from a lot of 4 good components and 3 defective components, it will contain, on average, 1.7 good components.

Moments of RVs: Expected Value

Theorem (Linearity of Expectations) *If a and b are constants then,*

$$\mu_{aX+b} = \mathbf{E}[aX + b] = a\mathbf{E}[X] + b$$

Proof:

$$\begin{aligned}\mu_{aX+b} &= \int (ax + b)p(X = x) dx \\ &= a \int xp(X = x) + b dx \\ &= a\mathbf{E}[X] + b\end{aligned}$$

Corollary Setting $a=0$ we see that $\mathbf{E}[b]=b$.

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Moments of RVs: Expected Value

Theorem (Linearity of Expectations) *For any finite collection of discrete RVs X_1, X_2, \dots, X_N with finite expectations,*

$$\mathbf{E} \left[\sum_{i=1}^N X_i \right] = \sum_{i=1}^N \mathbf{E}[X_i]$$

Note This holds for any collection of random variables. They do not need to be independent.

Expectations of Functions

Definition The expected value of a function $g(X)$ denoted by $\mathbf{E}[g(X)]$, is:

$$\mathbf{E}[g(X)] = \int g(x) p(X = x) dx$$

- The integral is over all values x
- Any function $g(X)$ of a random variable X is also a random variable
- Therefore, any function $g(X)$ has a probability distribution $p(g(X)=g)$

Example: Expected Value of a Function

Example 4.4: Suppose that the number of cars X that pass through a car wash between 4:00 P.M. and 5:00 P.M. on any sunny Friday has the following probability distribution:

x	4	5	6	7	8	9
$P(X = x)$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{6}$

Let $g(X) = 2X - 1$ represent the amount of money, in dollars, paid to the attendant by the manager. Find the attendant's expected earnings for this particular time period.

Example: Expected Value of a Function

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Let $g(X) = 2X - 1$ represent the amount of money, in dollars, paid to the attendant by the manager. Find the attendant's expected earnings for this particular time period.

Solution: By Theorem 4.1, the attendant can expect to receive

$$\begin{aligned} E[g(X)] &= E(2X - 1) = \sum_{x=4}^9 (2x - 1)f(x) \\ &= (7) \left(\frac{1}{12}\right) + (9) \left(\frac{1}{12}\right) + (11) \left(\frac{1}{4}\right) + (13) \left(\frac{1}{4}\right) \\ &\quad + (15) \left(\frac{1}{6}\right) + (17) \left(\frac{1}{6}\right) = \$12.67. \end{aligned}$$

Expected Value

Definition Consider a function of two random variables denoted,

$$g(X, Y)$$

The expected value of discrete X & Y is given by,

$$\mathbf{E}[g(X, Y)] = \sum_x \sum_y g(x, y) P(X = x, Y = y)$$

Or if X & Y are continuous,

$$\mathbf{E}[g(X, Y)] = \int \int g(x, y) P(X = x, Y = y) dx dy$$

Example

Example 4.6: Let X and Y be the random variables with joint probability distribution indicated in Table 3.1 on page 96. Find the expected value of $g(X, Y) = XY$. The table is reprinted here for convenience.

$f(x, y)$		x			Row
		0	1	2	Totals
y	0	$\frac{3}{28}$	$\frac{9}{28}$	$\frac{3}{28}$	$\frac{15}{28}$
	1	$\frac{3}{14}$	$\frac{3}{14}$	0	$\frac{3}{7}$
	2	$\frac{1}{28}$	0	0	$\frac{1}{28}$
Column Totals		$\frac{5}{14}$	$\frac{15}{28}$	$\frac{3}{28}$	1

Example

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	2	$\frac{1}{28}$	0	0	$\frac{1}{28}$
Column Totals		$\frac{5}{14}$	$\frac{15}{28}$	$\frac{3}{28}$	1

Solution: By Definition 4.2, we write

$$\begin{aligned} E(XY) &= \sum_{x=0}^2 \sum_{y=0}^2 xyf(x, y) \\ &= (0)(0)f(0, 0) + (0)(1)f(0, 1) \\ &\quad + (1)(0)f(1, 0) + (1)(1)f(1, 1) + (2)(0)f(2, 0) \\ &= f(1, 1) = \frac{3}{14}. \end{aligned}$$

Moments of RVs

Theorem: *If X and Y are independent then* $\mathbf{E}[XY] = \mathbf{E}[X]\mathbf{E}[Y]$

Proof:

$$\begin{aligned}\mathbf{E}[XY] &= \sum_x \sum_y (x \cdot y) p(X = x, Y = y) \\ &= \sum_x \sum_y (x \cdot y) p(X = x) p(Y = y) && \text{(Independence)} \\ &= \left(\sum_x x \cdot p(X = x) \right) \left(\sum_y y \cdot p(Y = y) \right) = \mathbf{E}[X]\mathbf{E}[Y] && \text{(Linearity of Expectation)}\end{aligned}$$

Example *Let $X_1, X_2 \in \{1, \dots, 6\}$ be RVs representing the result of rolling two fair standard die. **What is the mean of their product?***

$$\mathbf{E}[X_1 X_2] = \mathbf{E}[X_1] \mathbf{E}[X_2] = 3.5^2$$

Outline

- Mean of a Random Variable
- **Variance & Covariance of a Random Variable**
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Variance / Dispersion

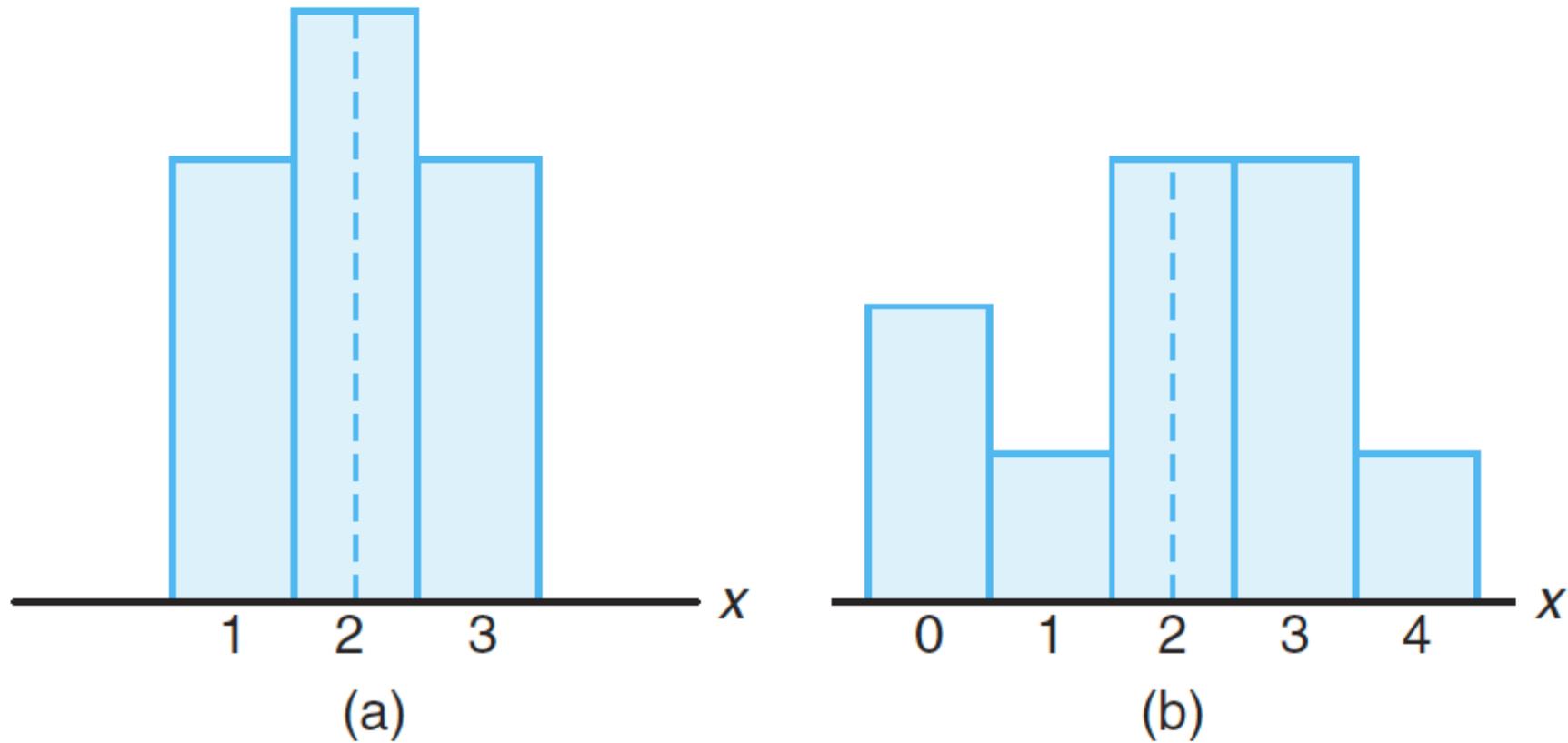


Figure 4.1: Distributions with equal means and unequal dispersions.

Moments of Discrete RVs: Variance

Definition The variance of a discrete RV X is defined as,

$$\sigma_X^2 = \mathbf{Var}[X] = \mathbf{E}[(X - \mathbf{E}[X])^2] \quad \boxed{\text{(X-units)}^2}$$

First compute the mean,

$$\mu = \mathbf{E}[X] = \sum_x P(X = x) \cdot x$$

Then compute variance as,

$$\sigma_X^2 = \mathbf{Var}[X] = \sum_x (x - \mu)^2 \cdot P(X = x)$$

The standard deviation is $\sigma_X = \sqrt{\mathbf{Var}[X]}$. (X-units)

Example: Variance

Example 4.8: Let the random variable X represent the number of automobiles that are used for official business purposes on any given workday. The probability distribution for company A [Figure 4.1(a)] is

x	1	2	3
$f(x)$	0.3	0.4	0.3

and that for company B [Figure 4.1(b)] is

x	0	1	2	3	4
$f(x)$	0.2	0.1	0.3	0.3	0.1

Show that the variance of the probability distribution for company B is greater than that for company A .

Example: Variance

Solution: For company A , we find that

$$\mu_A = E(X) = (1)(0.3) + (2)(0.4) + (3)(0.3) = 2.0,$$

and then

$$\sigma_A^2 = \sum_{x=1}^3 (x - 2)^2 = (1 - 2)^2(0.3) + (2 - 2)^2(0.4) + (3 - 2)^2(0.3) = 0.6.$$

For company B , we have

$$\mu_B = E(X) = (0)(0.2) + (1)(0.1) + (2)(0.3) + (3)(0.3) + (4)(0.1) = 2.0,$$

and then

$$\begin{aligned}\sigma_B^2 &= \sum_{x=0}^4 (x - 2)^2 f(x) \\ &= (0 - 2)^2(0.2) + (1 - 2)^2(0.1) + (2 - 2)^2(0.3) \\ &\quad + (3 - 2)^2(0.3) + (4 - 2)^2(0.1) = 1.6.\end{aligned}$$

Moments of Continuous RVs: Variance

Definition The variance of a continuous RV X is defined as,

$$\sigma_X^2 = \mathbf{Var}[X] = \mathbf{E}[(X - \mathbf{E}[X])^2] \quad \boxed{\text{(X-units)}^2}$$

First compute the mean,

$$\mu = \mathbf{E}[X] = \int x p(X = x) dx$$

Then compute variance as,

$$\sigma_X^2 = \mathbf{Var}[X] = \int (x - \mu)^2 p(X = x) dx$$

The standard deviation is $\sigma_X = \sqrt{\mathbf{Var}[X]}$. (X-units)

Moments of RVs

Lemma An equivalent form of variance is:

$$\sigma_X^2 = \mathbf{Var}[X] = \mathbf{E}[X^2] - (\mathbf{E}[X])^2$$

Proof Keep in mind that $E[X]$ is a constant,

$$\begin{aligned}\mathbf{E}[(X - \mathbf{E}[X])^2] &= \mathbf{E}[X^2 - 2X\mathbf{E}[X] + \mathbf{E}[X]^2] && \mathbf{(Distributive\ property)} \\ &= \mathbf{E}[X^2] - 2\mathbf{E}[X]\mathbf{E}[X] + \mathbf{E}[X]^2 && \mathbf{(Linearity\ of\ expectations)} \\ &= \mathbf{E}[X^2] - \mathbf{E}[X]^2 && \mathbf{(Algebra)}\end{aligned}$$

Example: Variance

Example 4.9: Let the random variable X represent the number of defective parts for a machine when 3 parts are sampled from a production line and tested. The following is the probability distribution of X .

x	0	1	2	3
$f(x)$	0.51	0.38	0.10	0.01

Using Theorem 4.2, calculate σ^2 .

Example: Variance

Example 4.9: Let the random variable X represent the number of defective parts for a machine when 3 parts are sampled from a production line and tested. The following is the probability distribution of X .

x	0	1	2	3
$f(x)$	0.51	0.38	0.10	0.01

Using Theorem 4.2, calculate σ^2 .

Solution: First, we compute

$$\mu = (0)(0.51) + (1)(0.38) + (2)(0.10) + (3)(0.01) = 0.61.$$

Now,

$$E(X^2) = (0)(0.51) + (1)(0.38) + (4)(0.10) + (9)(0.01) = 0.87.$$

Therefore,

$$\sigma^2 = 0.87 - (0.61)^2 = 0.4979.$$



Variance of a Function

Definition *The variance of a function $g(X)$ denoted by $\mathbf{Var}[g(X)]$ is:*

$$\sigma_{g(X)}^2 = \mathbf{Var}[g(X)] = \sum_x (g(x) - \mathbf{E}[g(X)])^2 \cdot P(X = x)$$

Example 4.11: Calculate the variance of $g(X) = 2X + 3$, where X is a random variable with probability distribution

x	0	1	2	3
$f(x)$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{2}$	$\frac{1}{8}$

Example: Variance of a Function

Solution: First, we find the mean of the random variable $2X + 3$. According to Theorem 4.1,

$$\mu_{2X+3} = E(2X + 3) = \sum_{x=0}^3 (2x + 3)f(x) = 6.$$

Now, using Theorem 4.3, we have

$$\begin{aligned}\sigma_{2X+3}^2 &= E\{[(2X + 3) - \mu_{2x+3}]^2\} = E[(2X + 3 - 6)^2] \\ &= E(4X^2 - 12X + 9) = \sum_{x=0}^3 (4x^2 - 12x + 9)f(x) = 4.\end{aligned}$$



Moments of RVs: Covariance

Definition The covariance of two RVs X and Y is defined as,

$$\sigma_{XY} = \mathbf{Cov}(X, Y) = \mathbf{E}[(X - \mathbf{E}[X])(Y - \mathbf{E}[Y])]$$

- Measures the relation between two random variables
- The *sign* of $\mathbf{Cov}(X, Y)$ indicates whether their relationship is positive or negative
- If X and Y are *independent* then $\mathbf{Cov}(X, Y) = 0$
- $\mathbf{Cov}(X, Y) = 0$ **does not** mean X and Y are independent

Moments of RVs: Covariance

Theorem The covariance of two RVs X and Y is given by,

$$\sigma_{XY} = \mathbf{Cov}(X, Y) = \mathbf{E}[XY] - \mathbf{E}[X]\mathbf{E}[Y]$$

Proof: For the discrete case, we can write

$$\begin{aligned}\sigma_{XY} &= \sum_x \sum_y (x - \mu_X)(y - \mu_Y)f(x, y) \\ &= \sum_x \sum_y xyf(x, y) - \mu_X \sum_x \sum_y yf(x, y) \\ &\quad - \mu_Y \sum_x \sum_y xf(x, y) + \mu_X \mu_Y \sum_x \sum_y f(x, y).\end{aligned}$$

Since

$$\mu_X = \sum_x xf(x, y), \quad \mu_Y = \sum_y yf(x, y), \quad \text{and} \quad \sum_x \sum_y f(x, y) = 1$$

for any joint discrete distribution, it follows that

$$\sigma_{XY} = E(XY) - \mu_X \mu_Y - \mu_Y \mu_X + \mu_X \mu_Y = E(XY) - \mu_X \mu_Y.$$

For the continuous case, the proof is identical with summations replaced by integrals. 

Example: Covariance

Example 4.13: Example 3.14 on page 95 describes a situation involving the number of blue refills X and the number of red refills Y . Two refills for a ballpoint pen are selected at random from a certain box, and the following is the joint probability distribution:

		x			$h(y)$
		0	1	2	
y	0	$\frac{3}{28}$	$\frac{9}{28}$	$\frac{3}{28}$	$\frac{15}{28}$
	1	$\frac{3}{14}$	$\frac{3}{14}$	0	$\frac{3}{7}$
	2	$\frac{1}{28}$	0	0	$\frac{1}{28}$
$g(x)$		$\frac{5}{14}$	$\frac{15}{28}$	$\frac{3}{28}$	1

Find the covariance of X and Y .

Example: Covariance

Solution: From Example 4.6, we see that $E(XY) = 3/14$. Now

$$\mu_X = \sum_{x=0}^2 xg(x) = (0) \left(\frac{5}{14}\right) + (1) \left(\frac{15}{28}\right) + (2) \left(\frac{3}{28}\right) = \frac{3}{4},$$

and

$$\mu_Y = \sum_{y=0}^2 yh(y) = (0) \left(\frac{15}{28}\right) + (1) \left(\frac{3}{7}\right) + (2) \left(\frac{1}{28}\right) = \frac{1}{2}.$$

Therefore,

$$\sigma_{XY} = E(XY) - \mu_X\mu_Y = \frac{3}{14} - \left(\frac{3}{4}\right) \left(\frac{1}{2}\right) = -\frac{9}{56}.$$



Moments of RVs

Lemma For any two RVs X and Y ,

$$\mathbf{Var}[X + Y] = \mathbf{Var}[X] + \mathbf{Var}[Y] + 2\mathbf{Cov}(X, Y)$$

e.g. variance is not a linear operator.

Proof

$$\mathbf{Var}[X + Y] = \mathbf{E}[(X + Y - \mathbf{E}[X + Y])^2]$$

(Linearity of expectation) $= \mathbf{E}[(X + Y - \mathbf{E}[X] - \mathbf{E}[Y])^2]$

(Distributive property) $= \mathbf{E}[(X - \mathbf{E}[X])^2 + (Y - \mathbf{E}[Y])^2 + 2(X - \mathbf{E}[X])(Y - \mathbf{E}[Y])]$

(Linearity of expectation) $= \mathbf{E}[(X - \mathbf{E}[X])^2] + \mathbf{E}[(Y - \mathbf{E}[Y])^2] + 2\mathbf{E}[(X - \mathbf{E}[X])(Y - \mathbf{E}[Y])]$

(Definition of Var / Cov) $= \mathbf{Var}[X] + \mathbf{Var}[Y] + 2\mathbf{Cov}(X, Y)$

Moments of RVs

Question: *What is the variance of the sum of independent RVs*

$$\begin{aligned}\mathbf{Var}[X_1 + X_2] &= \mathbf{Var}[X_1] + \mathbf{Var}[X_2] + 2\mathbf{Cov}(X_1, X_2) \\ &= \mathbf{Var}[X_1] + \mathbf{Var}[X_2] + 2\mathbf{E}[(X_1 - \mathbf{E}[X_1])(X_2 - \mathbf{E}[X_2])] \\ &= \mathbf{Var}[X_1] + \mathbf{Var}[X_2] + 2\mathbf{E}[(X_1 - \mathbf{E}[X_1])]\mathbf{E}[(X_2 - \mathbf{E}[X_2])] \\ &= \mathbf{Var}[X_1] + \mathbf{Var}[X_2] + 2(\mathbf{E}[X_1] - \mathbf{E}[X_1])(\mathbf{E}[X_2] - \mathbf{E}[X_2]) \\ &= \mathbf{Var}[X_1] + \mathbf{Var}[X_2]\end{aligned}$$

E.g. variance is a *linear operator* for independent RVs

Theorem: *If $X \perp Y$ then $\mathbf{Var}[X + Y] = \mathbf{Var}[X] + \mathbf{Var}[Y]$*

Corollary: *If $X \perp Y$ then $\mathbf{Cov}(X, Y) = 0$*

Variance of a Linear Function

Theorem *Let X and Y be random variables. The variance of the function $aX + bY + c$ is given by,*

$$\sigma_{aX+bY+c}^2 = \mathbf{Var}(aX + bY + c) = a^2 \mathbf{Var}(X) + b^2 \mathbf{Var}(Y) + 2ab \mathbf{Cov}(X, Y)$$

Corollary *Setting $b=0$ we see that,*

$$\mathbf{Var}(aX + c) = a^2 \mathbf{Var}(X)$$

Corollary *Setting $b=0$ and $c=0$ we see that,*

$$\mathbf{Var}(aX) = a^2 \mathbf{Var}(X)$$

So adding a constant c does not change the variance.

Example

Example 4.22: If X and Y are random variables with variances $\sigma_X^2 = 2$ and $\sigma_Y^2 = 4$ and covariance $\sigma_{XY} = -2$, find the variance of the random variable $Z = 3X - 4Y + 8$.

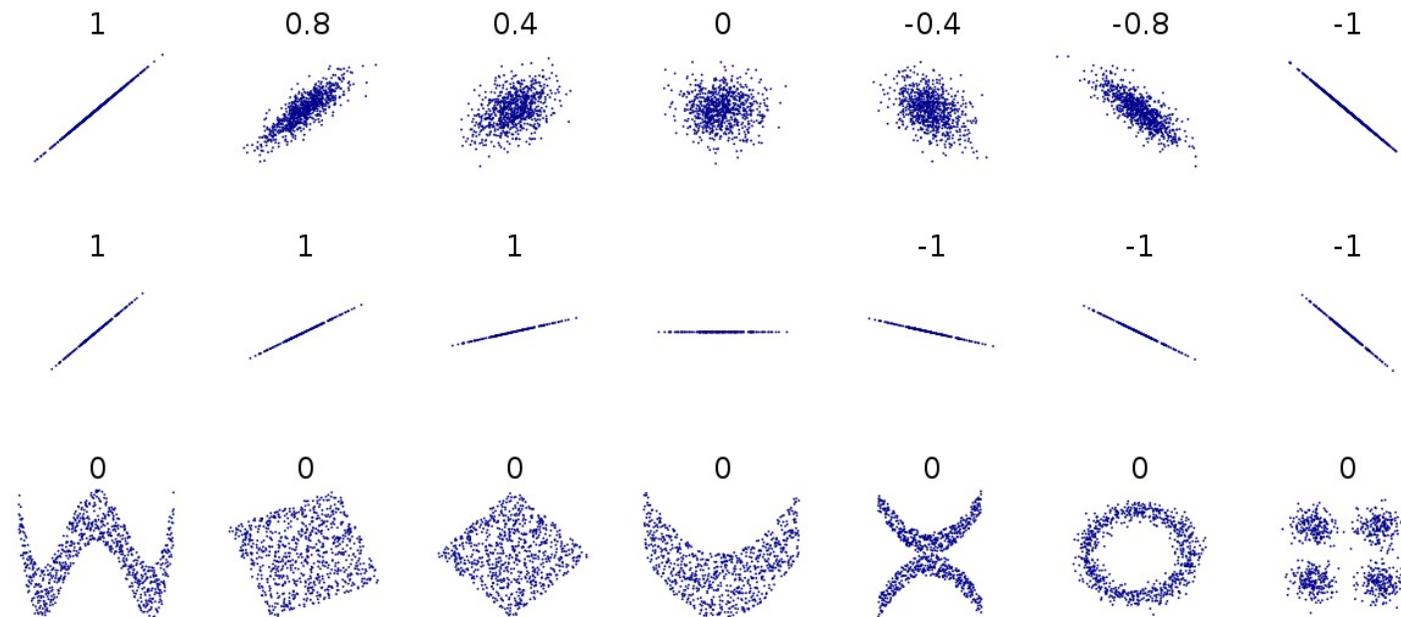
Solution:

$$\begin{aligned}\sigma_Z^2 &= \sigma_{3X-4Y+8}^2 = \sigma_{3X-4Y}^2 && \text{(by Corollary 4.6)} \\ &= 9\sigma_X^2 + 16\sigma_Y^2 - 24\sigma_{XY} && \text{(by Theorem 4.9)} \\ &= (9)(2) + (16)(4) - (24)(-2) = 130.\end{aligned}$$

Correlation

Definition *The correlation of two RVs X and Y is given by,*

$$\text{Corr}(X, Y) = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y} \quad \text{where} \quad \sigma_X = \sqrt{\text{Var}(X)}$$



Like covariance, only expresses linear relationships!

Example: Covariance

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	2	$\frac{1}{28}$	0	0	$\frac{1}{28}$
$g(x)$		$\frac{5}{14}$	$\frac{15}{28}$	$\frac{3}{28}$	1

Find the **correlation coefficient** between X and Y .

Example

Solution: Since

$$E(X^2) = (0^2) \left(\frac{5}{14} \right) + (1^2) \left(\frac{15}{28} \right) + (2^2) \left(\frac{3}{28} \right) = \frac{27}{28}$$

and

$$E(Y^2) = (0^2) \left(\frac{15}{28} \right) + (1^2) \left(\frac{3}{7} \right) + (2^2) \left(\frac{1}{28} \right) = \frac{4}{7},$$

we obtain

$$\sigma_X^2 = \frac{27}{28} - \left(\frac{3}{4} \right)^2 = \frac{45}{112} \text{ and } \sigma_Y^2 = \frac{4}{7} - \left(\frac{1}{2} \right)^2 = \frac{9}{28}.$$

Therefore, the correlation coefficient between X and Y is

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} = \frac{-9/56}{\sqrt{(45/112)(9/28)}} = -\frac{1}{\sqrt{5}}.$$

Outline

- Mean of a Random Variable
- Variance & Covariance of a Random Variable
- **Chebyshev's Theorem**

An Intuition About Variance

We expect a random variable with smaller variance to be more concentrated around the mean...

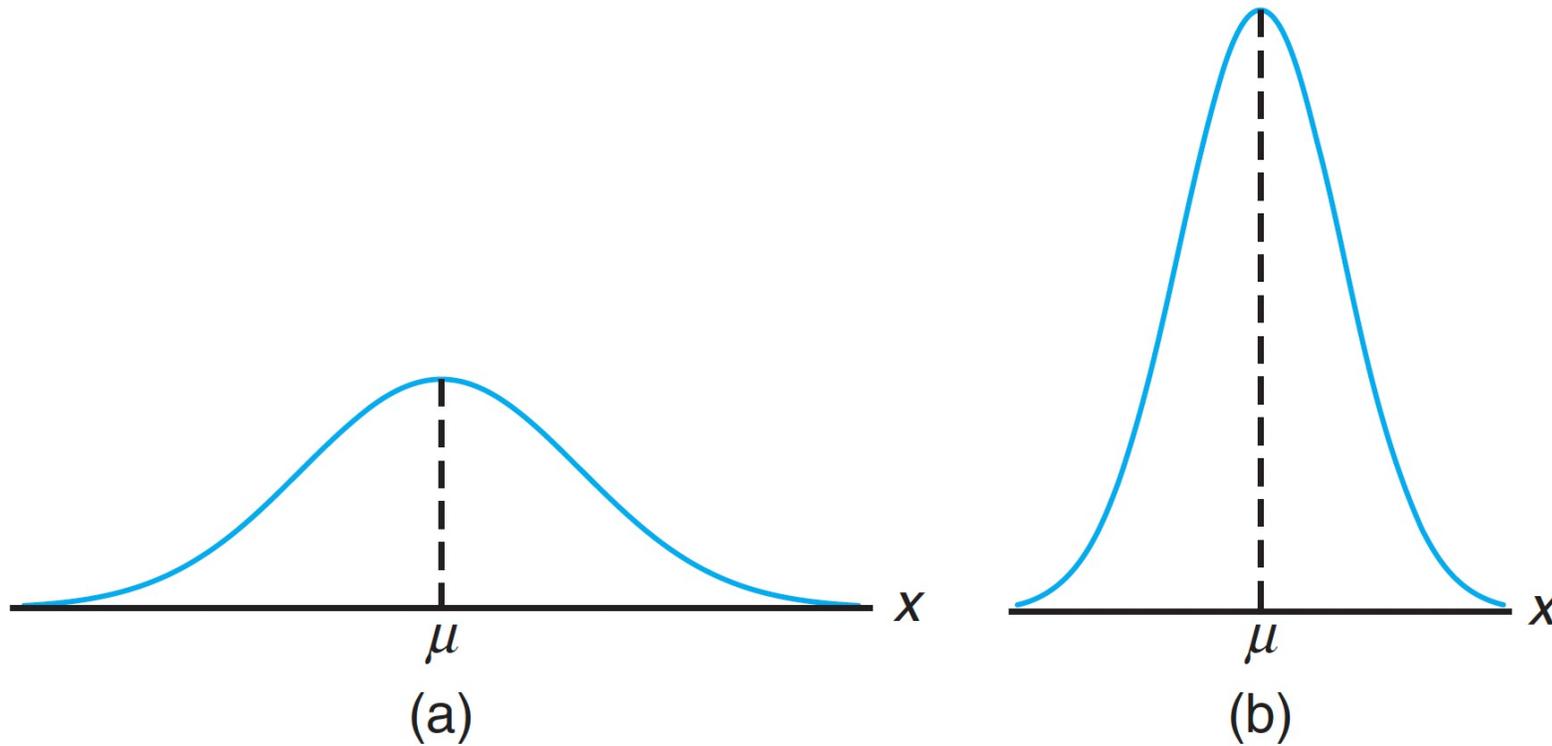


Figure 4.2: Variability of continuous observations about the mean.

Chebyshev's Theorem

Theorem 4.10: (Chebyshev's Theorem) The probability that any random variable X will assume a value within k standard deviations of the mean is at least $1 - 1/k^2$. That is,

$$P(\mu - k\sigma < X < \mu + k\sigma) \geq 1 - \frac{1}{k^2}.$$

- E.g. for $k=2$ the RV X has a probability of **at least** $1 - 1/2^2 = 3/4$ of falling within 2 standard deviations of the mean
- I.e. $3/4$ or more of the observations of **any distribution** lie in the interval $\mu \pm 2\sigma$

Example: Chebyshev's Theorem

Example 4.27: A random variable X has a mean $\mu = 8$, a variance $\sigma^2 = 9$, and an unknown probability distribution. Find

(a) $P(-4 < X < 20)$,

(b) $P(|X - 8| \geq 6)$.

Solution: (a) $P(-4 < X < 20) = P[8 - (4)(3) < X < 8 + (4)(3)] \geq \frac{15}{16}$.

(b) $P(|X - 8| \geq 6) = 1 - P(|X - 8| < 6) = 1 - P(-6 < X - 8 < 6)$
 $= 1 - P[8 - (2)(3) < X < 8 + (2)(3)] \leq \frac{1}{4}$.

Chebyshev's theorem makes no assumptions about the distribution $p(X)$, so the bounds can be rather weak.

