Computer Science

CSC580: Principles of Machine Learning

Probability and Statistics

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Outline

- Random Variables and Discrete Probability
- Fundamental Rules of Probability
- Expected Value and Moments
- Useful Discrete Distributions
- Continuous Probability

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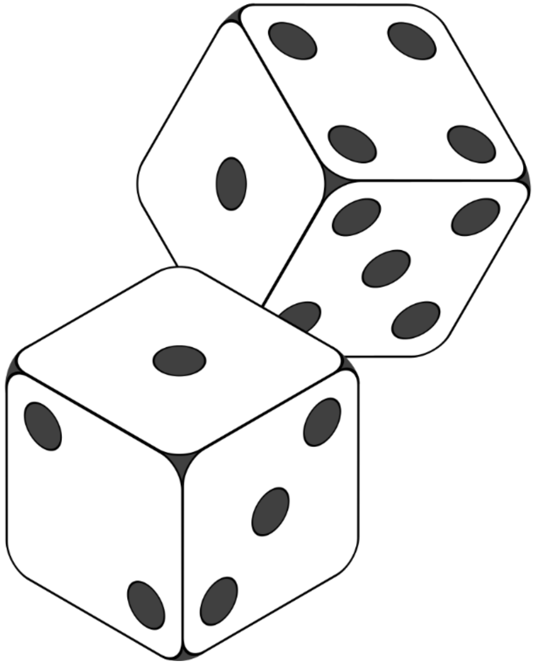
Random Events and Probability



Suppose we roll <u>two fair dice</u>…

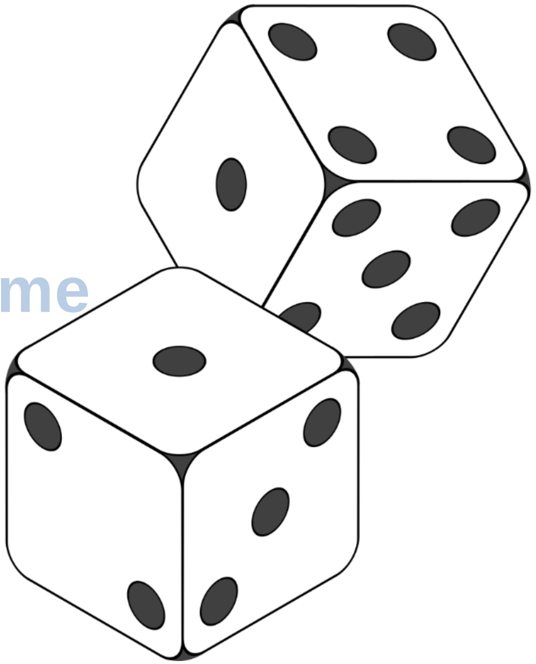
- ➢ What are the possible outcomes?
- ➢ What is the *probability* of rolling **even** numbers?
- ➢ What is the *probability* of rolling **odd** numbers?

***…probability theory** gives a mathematical formalism to addressing such questions…*

Definition An **experiment** or **trial** is any process that can be repeated with well-defined outcomes. It is *random* if more than one outcome is possible.

Random Events and Probability

Definition An **outcome** is a possible result of an experiment or trial, and the collection of all possible outcomes is the **sample space** of the experiment,



Outcome

Example (1,1), (1,2), …, (6,1), (6,2), …, (6,6)

Sample Space

Definition An **event** is a *set* of outcomes (a subset of the sample space),

Example Event Roll at least a single 1

{(1,1), (1,2), (1,3), …, (1,6), …, (6,1)}

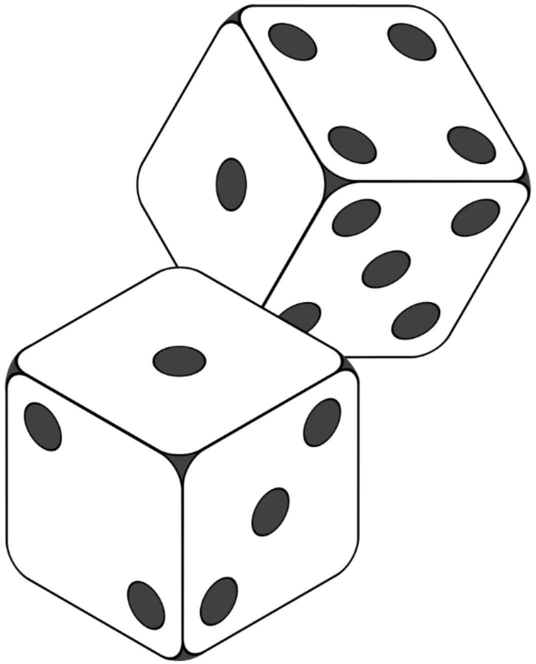
Random Events and Probability

Assume each outcome is equally likely, and sample space is finite, then the probability of event is:

$$P(E) = \frac{|E|}{|\Omega|}$$

Number of outcomes in event set

Number of possible outcomes in sample space



This is the **uniform probability distribution**

Example Probability that we roll *only* even numbers,

$$E^{\text{even}} = \{(2,2),(2,4),\ldots,(6,4),(6,6)\}$$

$$P(E^{\text{even}}) = \frac{|E^{\text{even}}|}{|\Omega|} = \frac{9}{36}$$

Random Events and Probability

Example Probability that the *sum of both dice* is even,

$$E^{\text{sum even}} = \{(1,1),(1,3),(1,5),\ldots,(2,2),(2,4),\ldots\}$$

$$P(E^{\text{sum even}}) = \frac{|E^{\text{sum even}}|}{|\Omega|} = \frac{18}{36} = \frac{1}{2}$$

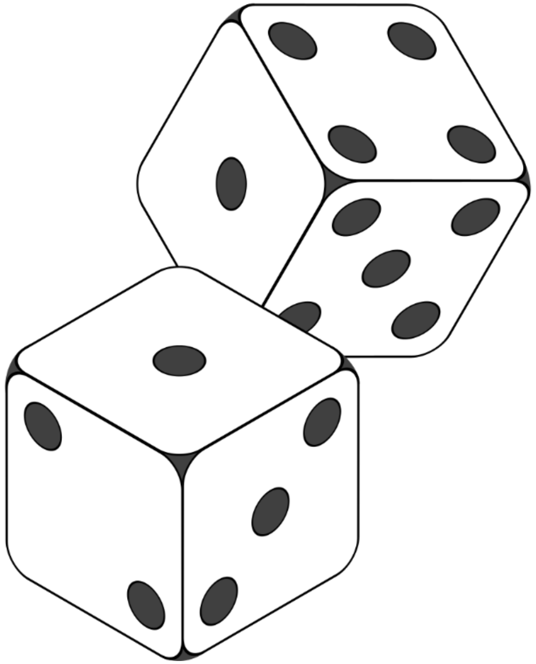
Example Probability that the *sum of both dice* is greater than 12,

$$E^{>12} = \emptyset$$

$$P(E^{>12}) = \frac{|E^{>12}|}{|\Omega|} = 0$$

i.e. we can reason about the probability of impossible outcomes

Random Variables



Suppose we are interested in a distribution over the sum of dice…

Option 1 Let E_i be event that the sum equals *i*

Two dice example:

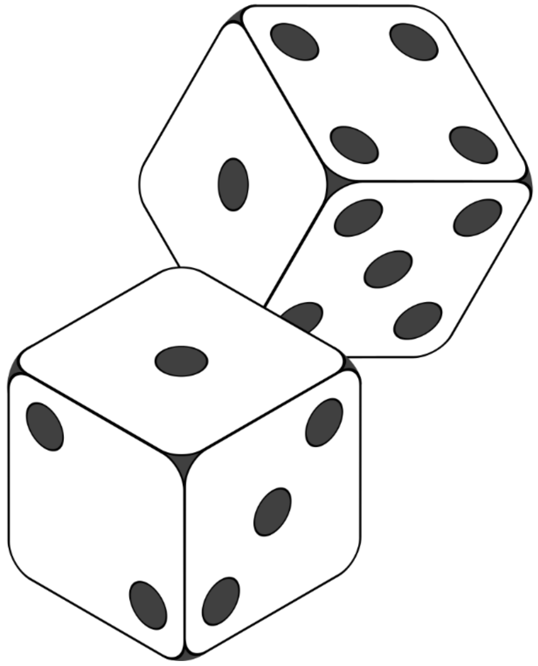
$$E_2 = \{(1,1)\} \quad E_3 = \{(1,2),(2,1)\} \quad E_4 = \{(1,3),(2,2),(3,1)\}$$

$$E_5 = \{(1,4),(2,3),(3,2),(4,1)\} \quad E_6 = \{(1,5),(2,4),(3,3),(4,2),(5,1)\}$$

Enumerate all possible means of obtaining desired sum. Gets cumbersome for N>2 dice…

Random Variables

<u>Option 2</u> Use a function of sample space…



(Informally) A random variable is an unknown quantity that maps events to numeric values.

Example X is the *sum of two dice* with values,

$$X \in \{2, 3, 4, \ldots, 12\}$$

Example Flip a coin and let random variable Y represent the outcome,

$$Y \in \{\text{Heads}, \text{Tails}\}$$

Discrete vs. Continuous Probability

Discrete RVs take on a finite or countably infinite set of values

Continuous RVs take an uncountably infinite set of values

- Representing / interpreting / computing probabilities becomes more complicated in the continuous setting

- We will focus on discrete RVs for now…

Random Variables and Probability

Capitol letters represent random variables

Lowercase letters are realized *values*

$$X = x$$

$X = x$ is the **event** that X takes the value x

Example Let X be the random variable (RV) representing the sum of two dice with values,

$$X \in \{2, 3, 4, \ldots, 12\}$$

X=5 is the *event* that the dice sum to 5.

Probability Mass Function

A function $p(X)$ is a **probability mass function (PMF)** of a discrete random variable if the following conditions hold:

(a) It is nonnegative for all values in the support,

$$p(X = x) \geq 0$$

(b) The sum over all values in the support is 1,

$$\sum_x p(X = x) = 1$$

Intuition Probability mass is conserved, just as in physical mass. Reducing probability mass of one event must increase probability mass of other events so that the definition holds...

Probability Mass Function

Example Let X be the outcome of a single fair die. It has the PMF,

$$p(X = x) = \frac{1}{6} \qquad \text{for } x = 1, \ldots, 6$$

Uniform Distribution

Example We can often represent the PMF as a vector. Let S be an RV that is the *sum of two fair dice*. The PMF is then,

$$p(S) = \begin{pmatrix} p(S = 2) \\ p(S = 3) \\ p(S = 4) \\ \vdots \\ p(S = 12) \end{pmatrix} = \begin{pmatrix} 1/36 \\ 1/18 \\ 1/2 \\ \vdots \\ 1/36 \end{pmatrix}$$

Observe that S does <u>not</u> follow a uniform distribution

Functions of Random Variables

Any function $f(X)$ of a random variable X is also a random variable and it has a probability distribution

Example Let X_1 be an RV that represents the result of a fair die, and let X_2 be the result of another fair die. Then,

$$S = X_1 + X_2$$

Is an RV that is the *sum of two fair dice* with PMF $p(S)$.

NOTE Even if we know the PMF $p(X)$ and we know that the PMF $p(f(X))$ exists, it is not always easy to calculate!

PMF Notation

- We use $p(X)$ to refer to the probability mass *function* (i.e. a function of the RV X)

- We use $p(X=x)$ to refer to the probability of the *outcome* $X=x$ (also called an “event”)

- We will often use $p(x)$ as shorthand for $p(X=x)$

Joint Probability

Definition Two (discrete) RVs X and Y have a *joint PMF* denoted by $p(X,Y)$ and the probability of the event X=x and Y=y denoted by $p(X=x,Y=y)$ where,

(a) It is nonnegative for all values in the support,

$$p(X=x,Y=y) \geq 0$$

(b) The sum over all values in the support is 1,

$$\sum_x \sum_y p(X=x,Y=y) = 1$$

Joint Probability

Let X and Y be *binary RVs.* We can represent the joint PMF p(X,Y) as a 2x2 array (table):

		Y	
		0	1
X	0	0.04	0.36
	1	0.30	0.30

All values are nonnegative

Joint Probability

Let X and Y be *binary RVs.* We can represent the joint PMF p(X,Y) as a 2x2 array (table):

		Y = 0	Y = 1
X	0	0.04	0.36
	1	0.30	0.30

The sum over all values is 1:
0.04 + 0.36 + 0.30 + 0.30 = 1

Joint Probability

Let X and Y be *binary RVs.* We can represent the joint PMF p(X,Y) as a 2x2 array (table):

		Y = 0	Y = 1
X	0	0.04	0.36
	1	0.30	0.30

P(X=1, Y=0) = 0.30

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Fundamental Rules of Probability

Given two RVs X and Y the **conditional distribution** is:

$$p(X \mid Y) = \frac{p(X,Y)}{p(Y)} = \frac{p(X,Y)}{\sum_x p(X{=}x,Y)}$$

Multiply both sides by $p(Y)$ to obtain the **probability chain rule**:

$$p(X, Y) = p(Y)p(X \mid Y)$$

The probability chain rule extends to N RVs $X_1, X_2, \ldots, X_N$:

$$p(X_1, X_2, \ldots, X_N) = p(X_1)p(X_2 \mid X_1) \ldots p(X_N \mid X_{N-1}, \ldots, X_1)$$

$$= p(X_1) \prod_{i=2}^{N} p(X_i \mid X_{i-1}, \ldots, X_1)$$

Chain rule valid for any ordering

Fundamental Rules of Probability

Law of total probability

$$p(Y) = \sum_x p(Y, X = x)$$

- P(y) is a **marginal** distribution
- This is called **marginalization**

Proof

$$\sum_x p(Y, X = x) = \sum_x p(Y) p(X = x \mid Y) \quad \text{(chain rule)}$$

$$= p(Y) \sum_x p(X = x \mid Y) \quad \text{(distributive property)}$$

$$= p(Y) \quad \text{(PMF sums to 1)}$$

Generalization for conditionals:

$$p(Y \mid Z) = \sum_x p(Y, X = x \mid Z)$$

Tabular Method

Let X, Y be binary RVs with the joint probability table

For Binomial use K-by-K probability table.

		Y: y_1	Y: y_2	$P(x)$
X	x_1	0.04	0.36	0.4 ← $P(x_1)$
X	x_2	0.30	0.30	0.6 ← $P(x_2)$
$P(y)$		0.34 ← $P(y_1)$	0.66 ← $P(y_2)$	

$P(y_1)=P(x_1,y_1)+P(x_2,y_1)$
$P(y_2)=P(x_1,y_2)+P(x_2,y_2)$
[i.e., sum down columns]

$P(x_1)=P(x_1,y_1)+P(x_1,y_2)$
$P(x_2)=P(x_2,y_1)+P(x_2,y_2)$
[i.e., sum across rows]

Tabular Method

We don't care about event $Y=y_2$

		Y: y_1	Y: y_2
X	x_1	0.04	Censored!
X	x_2	0.30	Censored!
		0.34	

0.34 ← $P(y_1)$

$P(x|y_1)=?$

Tabular Method

		$Y=y_1$
X	x_1	0.04
	x_2	0.30

0.34

$P(y_1)$

→

0.04 / 0.34

0.30 / 0.34

$P(x|y_1)$

These sum to one:
A conditional probability distribution is still a probability distribution

Intuition Check

<u>Question:</u> Roll two dice and let their outcomes be $X_1, X_2 \in \{1, \ldots, 6\}$ *for die 1 and die 2, respectively. Recall the definition of conditional probability,*

$$p(X_1 \mid X_2) = \frac{p(X_1, X_2)}{p(X_2)}$$

Which of the following are true?

a) $p(X_1 = 1 | X_2 = 1) > p(X_1 = 1)$

b) $p(X_1 = 1 | X_2 = 1) = p(X_1 = 1)$ Outcome of die 2 doesn't *affect* die 1

c) $p(X_1 = 1 | X_2 = 1) < p(X_1 = 1)$

Intuition Check

<u>*Question:*</u> *Let* $X_1 \in \{1, \ldots, 6\}$ *be outcome of die 1, as before. Now let* $X_3 \in \{2, 3, \ldots, 12\}$ *be the* sum of both dice. Which of the following are true?

- a) $p(X_1 = 1 | X_3 = 3) > p(X_1 = 1)$
- b) $p(X_1 = 1 | X_3 = 3) = p(X_1 = 1)$
- c) $p(X_1 = 1 | X_3 = 3) < p(X_1 = 1)$

Only 2 ways to get $X_3 = 3$ *, each with equal probability:*

$(X_1 = 1, X_2 = 2)$ *or* $(X_1 = 2, X_2 = 1)$

so

$$p(X_1 = 1 \mid X_3 = 3) = \frac{1}{2} > \frac{1}{6} = p(X_1 = 1)$$

Dependence of RVs

Intuition…

Consider $P(B|A)$ where you want to bet on B

Should you pay to know A?

In general you would pay something for A if it changed your belief about B. In other words if,

$$P(B|A) \neq P(B)$$

Independence of RVs

Definition *Two random variables* X *and* Y *are* <u>*independent*</u> *if and only if,*

$$p(X = x, Y = y) = p(X = x)p(Y = y)$$

for all values x and y, and we say $X \perp Y$.

Definition RVs $X_1, X_2, \ldots, X_N$ are <u>mutually independent</u> if and only if,

$$p(X_1 = x_1, \ldots, X_N = x_N) = \prod_{i=1}^{N} p(X_i = x_i)$$

- Independence is *symmetric:* $X \perp Y \Leftrightarrow Y \perp X$
- Equivalent definition of independence: $p(X \mid Y) = p(X)$

Independence of RVs

Definition *Two random variables X and Y are <u>conditionally independent</u> given Z if and only if,*

$$p(X = x, Y = y \mid Z = z) = p(X = x \mid Z = z)p(Y = y \mid Z = z)$$

for all values x, y, and z, and we say that $X \perp Y \mid Z$.

- N RVs conditionally independent, given Z, if and only if:

$$p(X_1, \ldots, X_N \mid Z) = \prod_{i=1}^{N} p(X_i \mid Z)$$

Shorthand notation
Implies for all *x, y, z*

- Equivalent def'n of conditional independence: $p(X \mid Y, Z) = p(X \mid Z)$
- Symmetric: $X \perp Y \mid Z \Leftrightarrow Y \perp X \mid Z$

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Moments of RVs

Definition *The* *expectation* *of a discrete RV* X*, denoted by* $\mathbf{E}[X]$*, is:*

$$\mathbf{E}[X] = \sum_x x\, p(X = x)$$

Summation over all values in domain of X

Example Let X be the sum of two fair dice, then:

$$\mathbf{E}[X] = \frac{1}{36} \cdot 2 + \frac{1}{18} \cdot 3 + \ldots + \frac{1}{36} \cdot 12 = 7$$

Theorem (Linearity of Expectations) *For any finite collection of discrete RVs* $X_1, X_2, \ldots, X_N$ *with finite expectations,*

Corollary *For any constant c*
$\mathbf{E}[cX] = c\mathbf{E}[X]$

$$\mathbf{E}\left[\sum_{i=1}^N X_i\right] = \sum_{i=1}^N \mathbf{E}[X_i]$$

E.g. for two RVs X and Y
$\mathbf{E}[X+Y] = \mathbf{E}[X] + \mathbf{E}[Y]$

Moments of RVs

Theorem: *If* $X \perp Y$ *then* $\mathbf{E}[XY] = \mathbf{E}[X]\mathbf{E}[Y]$ *.*

Proof:

$$\mathbf{E}[XY] = \sum_x \sum_y (x \cdot y) p(X = x, Y = y)$$

$$= \sum_x \sum_y (x \cdot y) p(X = x) p(Y = y)$$ **(Independence)**

$$= \left(\sum_x x \cdot p(X = x) \right) \left(\sum_y y \cdot p(Y = y) \right) = \mathbf{E}[X]\mathbf{E}[Y]$$ **(Linearity of Expectation)**

Example *Let* $X_1, X_2 \in \{1, \ldots, 6\}$ *be RVs representing the result of rolling two fair standard die.* *What is the mean of their product?*

$$\mathbf{E}[X_1 X_2] = \mathbf{E}[X_1]\mathbf{E}[X_2] = 3.5^2$$

Moments of RVs

Definition *The <u>conditional expectation</u> of a discrete RV* X*, given* Y *is:*

$$\mathbf{E}[X \mid Y = y] = \sum_x x\, p(X = x \mid Y = y)$$

Example Roll two standard six-sided dice and let X be the result of the first die and let Y be the sum of both dice, then:

$$\mathbf{E}[X_1 \mid Y = 5] = \sum_{x=1}^{4} x\, p(X_1 = x \mid Y = 5)$$

$$= \sum_{x=1}^{4} x \frac{p(X_1 = x, Y = 5)}{p(Y = 5)} = \sum_{x=1}^{4} x \frac{1/36}{4/36} = \frac{5}{2}$$

Conditional expectation follows properties of expectation (linearity, etc.)

Moments of RVs

Law of Total Expectation *Let X and Y be discrete RVs with finite expectations, then:*

$$\mathbf{E}[X] = \mathbf{E}_Y[\mathbf{E}_X[X \mid Y]]$$

Proof

$$\mathbf{E}_Y[\mathbf{E}_X[X \mid Y]] = \mathbf{E}_Y\left[\sum_x x \cdot p(x \mid Y)\right]$$

$$= \sum_y \left[\sum_x x \cdot p(x \mid y)\right] \cdot p(y)$$ **(Definition of expectation)**

$$= \sum_y \sum_x x \cdot p(x, y)$$ **(Probability chain rule)**

$$= \sum_x x \sum_y \cdot p(x, y)$$ **(Linearity of expectations)**

$$= \sum_x x \cdot p(x) = \mathbf{E}[X]$$ **(Law of total probability)**

Moments of RVs

Definition *The* <u>*variance*</u> *of a RV* X *is defined as,*

$$\mathbf{Var}[X] = \mathbf{E}[(X - \mathbf{E}[X])^2]$$ (X-units)^2

The <u>*standard deviation*</u> *is* $\sigma[X] = \sqrt{\mathbf{Var}[X]}$. (X-units)

Lemma An equivalent form of variance is:

$$\mathbf{Var}[X] = \mathbf{E}[X^2] - (\mathbf{E}[X])^2$$

Proof Keep in mind that $E[X]$ is a constant,

$$\mathbf{E}[(X - \mathbf{E}[X])^2] = \mathbf{E}[X^2 - 2X\mathbf{E}[X] + \mathbf{E}[X]^2]$$ **(Distributive property)**

$$= \mathbf{E}[X^2] - 2\mathbf{E}[X]\mathbf{E}[X] + \mathbf{E}[X]^2$$ **(Linearity of expectations)**

$$= \mathbf{E}[X^2] - \mathbf{E}[X]^2$$ **(Algebra)**

Moments of RVs

Definition *The covariance of two RVs* X *and* Y *is defined as,*

$$\mathbf{Cov}(X,Y) = \mathbf{E}[(X - \mathbf{E}[X])(Y - \mathbf{E}[Y])]$$

Lemma *For any two RVs* X *and* Y*,*

$$\mathbf{Var}[X+Y] = \mathbf{Var}[X] + \mathbf{Var}[Y] + 2\mathbf{Cov}(X,Y)$$

e.g. variance is not a linear operator.

Proof $\mathbf{Var}[X+Y] = \mathbf{E}[(X+Y-\mathbf{E}[X+Y])^2]$

(Linearity of expectation) $= \mathbf{E}[(X+Y-\mathbf{E}[X]-\mathbf{E}[Y])^2]$

(Distributive property) $= \mathbf{E}[(X-\mathbf{E}[X])^2 + (Y-\mathbf{E}[Y])^2 + 2(X-\mathbf{E}[X])(Y-\mathbf{E}[Y])]$

(Linearity of expectation) $= \mathbf{E}[(X-\mathbf{E}[X])^2] + \mathbf{E}[(Y-\mathbf{E}[Y])^2] + 2\mathbf{E}[(X-\mathbf{E}[X])(Y-\mathbf{E}[Y])]$

(Definition of Var / Cov) $= \mathbf{Var}[X] + \mathbf{Var}[Y] + 2\mathbf{Cov}(X,Y)$

Moments of RVs

Question: *What is the variance of the sum of independent RVs*

$$\begin{aligned}\mathbf{Var}[X_1 + X_2] &= \mathbf{Var}[X_1] + \mathbf{Var}[X_2] + 2\mathbf{Cov}(X_1, X_2) \\ &= \mathbf{Var}[X_1] + \mathbf{Var}[X_2] + 2\mathbf{E}[(X_1 - \mathbf{E}[X_1])(X_2 - \mathbf{E}[X_2])] \\ &= \mathbf{Var}[X_1] + \mathbf{Var}[X_2] + 2\mathbf{E}[(X_1 - \mathbf{E}[X_1])]\mathbf{E}[(X_2 - \mathbf{E}[X_2])] \\ &= \mathbf{Var}[X_1] + \mathbf{Var}[X_2] + 2\left(\mathbf{E}[X_1] - \mathbf{E}[X_1]\right)\left(\mathbf{E}[X_2] - \mathbf{E}[X_2]\right) \\ &= \mathbf{Var}[X_1] + \mathbf{Var}[X_2]\end{aligned}$$

E.g. variance is a *linear operator* for independent RVs

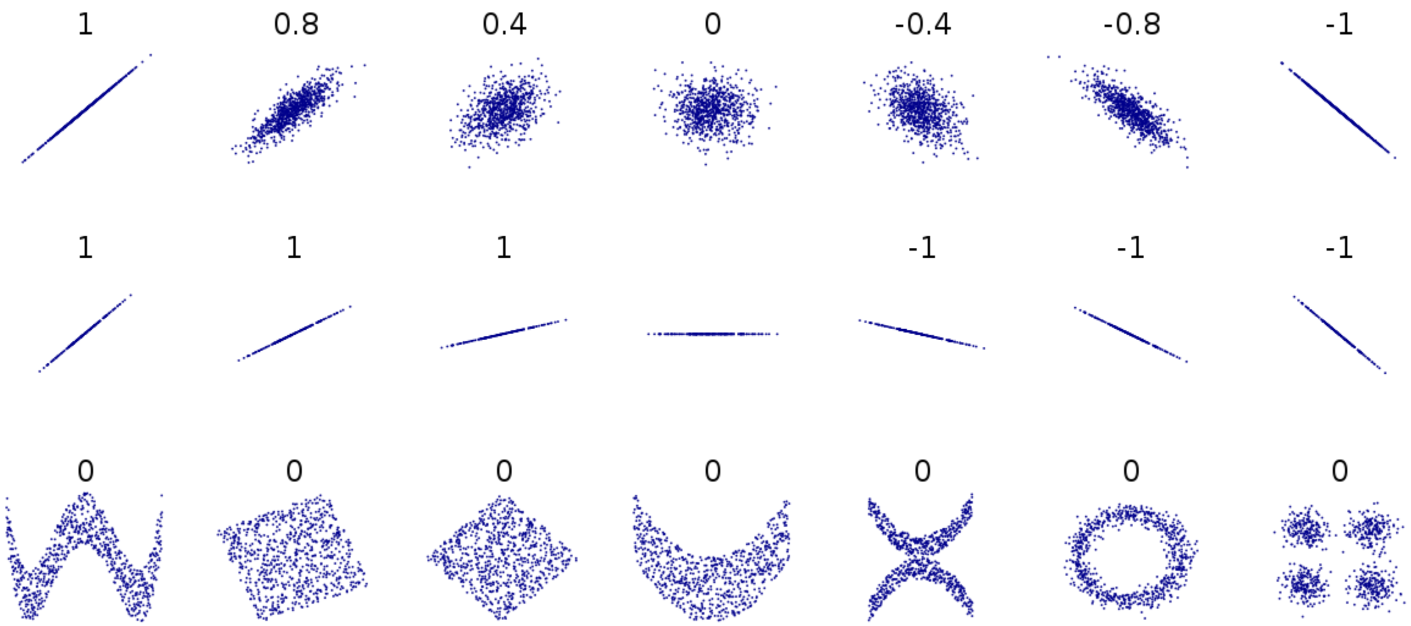
Theorem: *If* $X \perp Y$ *then* $\mathbf{Var}[X + Y] = \mathbf{Var}[X] + \mathbf{Var}[Y]$

Corollary: *If* $X \perp Y$ *then* $\mathbf{Cov}(X, Y) = 0$

Correlation

Definition *The correlation of two RVs X and Y is given by,*

$$\mathbf{Corr}(X,Y) = \frac{\mathbf{Cov}(X,Y)}{\sigma_X \sigma_Y} \quad \textit{where} \quad \sigma_X = \sqrt{\mathbf{Var}(X)}$$



Like covariance, only expresses linear relationships!

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Useful Discrete Distributions

Bernoulli *A.k.a. the **coinflip** distribution on <u>binary</u> RVs* $X \in \{0, 1\}$

$$p(X) = \pi^X (1 - \pi)^{(1-X)}$$

Where π *is the probability of **success** (e.g. heads), and also the mean*

$$\mathbf{E}[X] = \pi \cdot 1 + (1 - \pi) \cdot 0 = \pi$$

Suppose we flip N <u>independent</u> coins $X_1, X_2, \ldots, X_N$, what is the distribution over their sum $Y = \sum_{i=1}^N X_i$

Num. "successes" out of N trials

Num. ways to obtain k successes out of N

Binomial Dist. $$p(Y = k) = \binom{N}{k} \pi^k (1 - \pi)^{N-k}$$

Binomial Mean: $\mathbf{E}[Y] = N \cdot \pi$ Sum of means for N indep. Bernoulli RVs

Useful Discrete Distributions

Question: How many flips until we observe a success?

Geometric *Distribution on number of independent draws of* $X \sim \text{Bernoulli}(\pi)$ *until success:*

$$p(Y = n) = (1-\pi)^{n-1}\pi \qquad \mathbf{E}[Y] = \frac{1}{\pi}$$

E.g. for fair coin $\pi = 1/2$ takes two flips on avg.

e.g. there must be n-1 failures (tails) before a success (heads).

Question: How many more flips if we have already seen k failures?

$$p(Y = n + k \mid Y > k) = \frac{p(Y=n+k, Y>k)}{p(Y>k)} = \frac{p(Y=n+k)}{p(Y>k)}$$

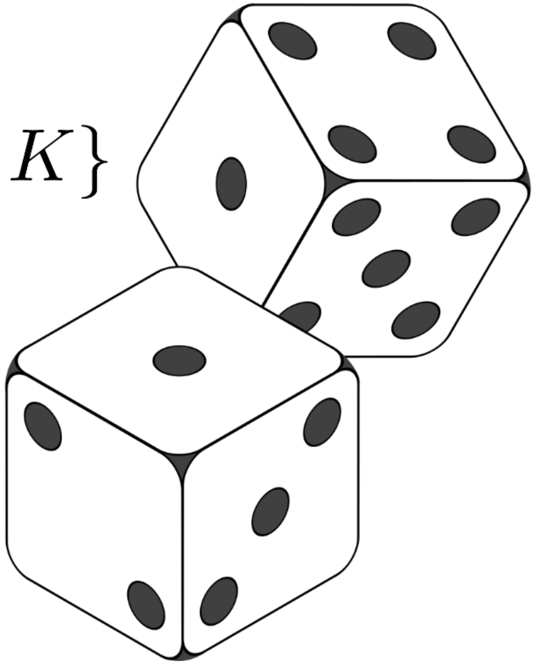
$$= \frac{(1-\pi)^{n+k-1}\pi}{\sum_{i=k}^{\infty}(1-\pi)^i\pi} = \frac{(1-\pi)^{n+k-1}\pi}{(1-\pi)^k} = (1-\pi)^{n-1}\pi = p(Y = n)$$

For $0 < x < 1, \sum_{i=k}^{\infty} x^i = x^k/(1-x)$

Corollary: $p(Y > k) = (1-\pi)^{k-1}$

Useful Discrete Distributions

Categorical *Distribution on integer-valued RV* $X \in \{1, \ldots, K\}$



$$p(X) = \prod_{k=1}^{K} \pi_k^{\mathbf{I}(X=k)} \quad \text{or} \quad p(X) = \sum_{k=1}^{K} \mathbf{I}(X=k) \cdot \pi_k$$

with parameter $p(X=k) = \pi_k$ *and Kroenecker delta:*

$$\mathbf{I}(X=k) = \begin{cases} 1, & \text{If } X = k \\ 0, & \text{Otherwise} \end{cases}$$

Can also represent X as *one-hot* binary vector,

$$X \in \{0,1\}^K \quad \text{where} \quad \sum_{k=1}^{K} X_k = 1 \quad \text{then} \quad p(X) = \prod_{k=1}^{K} \pi_k^{X_k}$$

This representation is special case of the **multinomial distribution**

Useful Discrete Distributions

What if we count outcomes of *N* independent categorical RVs?

Multinomial *Distribution on K-vector* $X \in \{0, N\}^K$ *of counts of N repeated trials* $\sum_{k=1}^K X_k = N$ *with PMF:*

$$p(x_1, \ldots, x_K) = \binom{n}{x_1 x_2 \ldots x_K} \prod_{k=1}^K \pi_k^{x_k}$$

Number of ways to partition N objects into K groups:

$$\binom{n}{x_1 x_2 \ldots x_K} = \frac{n!}{x_1! x_2! \ldots x_K!}$$

Leading term ensures PMF is properly normalized:

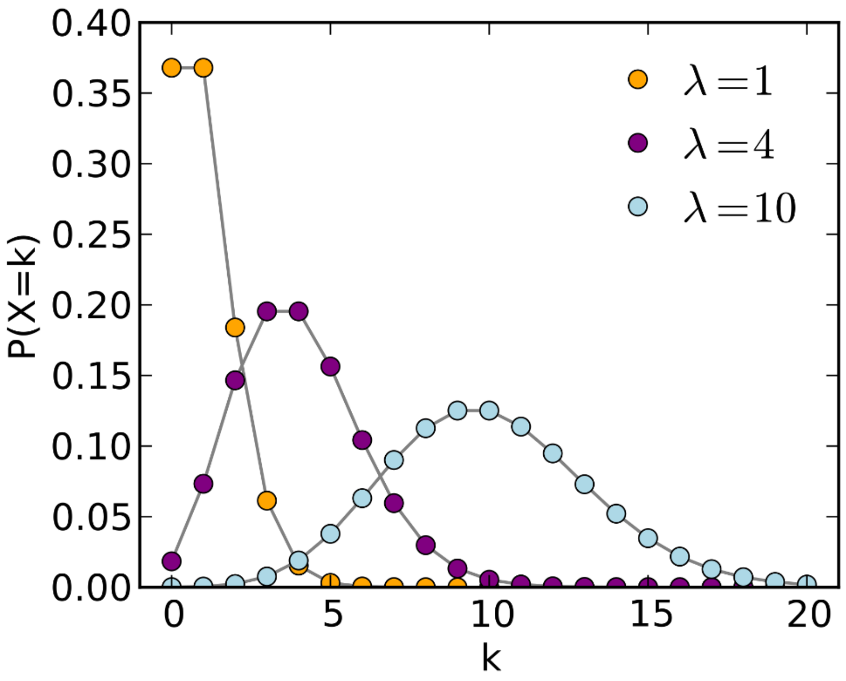
$$\sum_{x_1} \sum_{x_2} \cdots \sum_{x_K} p(x_1, x_2, \ldots, x_K) = 1$$

Useful Discrete Distributions

*A **Poisson** RV X with rate parameter λ has the following distribution:*

Mean and variance both scale with parameter

$$p(X = k) = \frac{e^{-\lambda}\lambda^k}{k!} \qquad \mathbf{E}[X] = \mathbf{Var}[X] = \lambda$$



Represents number of times an *event* occurs in an interval of time or space.

***Ex.** Probability of overflow floods in 100 years,*

$$p(k \text{overflow floods in 100 yrs}) = \frac{e^{-1}1^k}{k!}$$

Avg. 1 overflow flood every 100 years, makes setting rate parameter easy.

***Lemma (additive closure)** The sum of a finite number of Poisson RVs is a Poisson RV.*

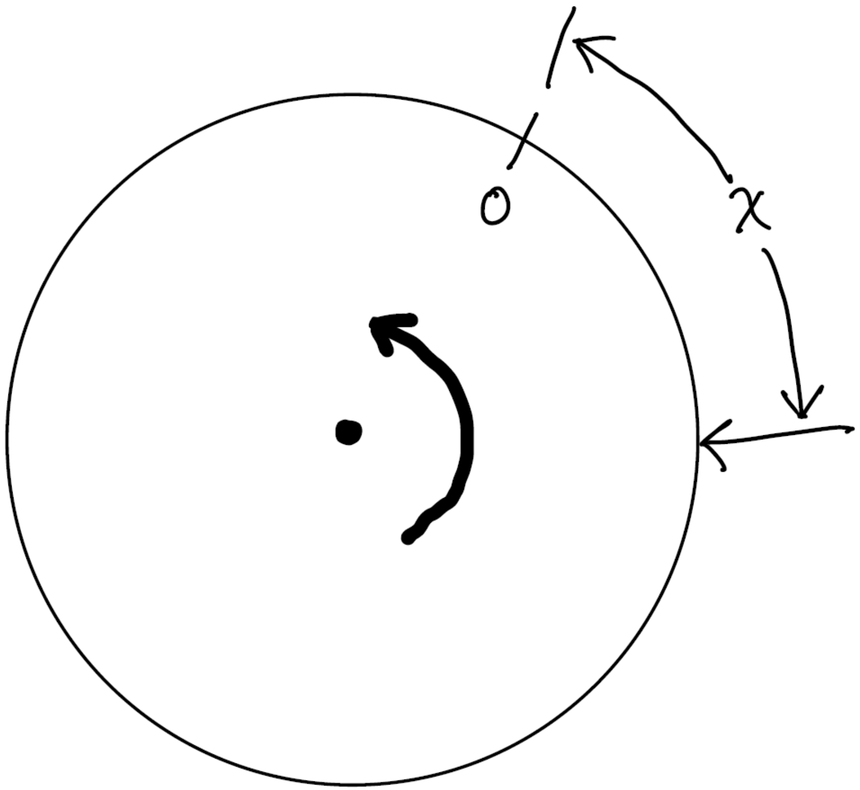
$$X \sim \text{Poisson}(\lambda_1), \quad Y \sim \text{Poisson}(\lambda_2), \quad X + Y \sim \text{Poisson}(\lambda_1 + \lambda_2)$$

Outline

- Random Variables and Discrete Probability
- Fundamental Rules of Probability
- Expected Value and Moments
- Useful Discrete Distributions
- Continuous Probability

Continuous Probability

Experiment Spin continuous wheel and measure X displacement from 0



Question Assuming uniform probability, what is $p(X = x)$?

Continuous Probability

- Let $p(X = x) = \pi$ be the probability of any single outcome

- Let $S(k)$ be set of any k *distinct* points in $[0, 1)$ then,

$$P(x \in S(k)) = k\pi$$

- Since $0 < P(x \in S(k)) < 1$ we have that $k\pi < 1$ for any k

- Therefore: $\pi = 0$ and $P(x \in S(k)) = p(X = x) = 0$

Continuous Probability

- We have a well-defined event that x takes a value in set $x \in S(k)$
- Clearly this event can happen… i.e. **it is possible**
- But we have shown it has zero probability of occurring,

$$P(x \in S(k)) = 0$$

- The probability that it **doesn't happen** is,

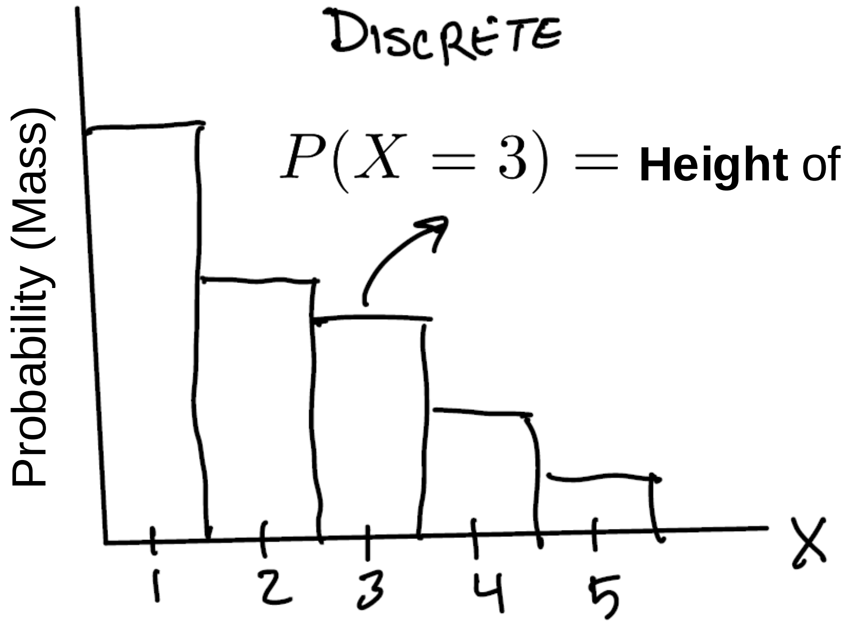
$$P(x \notin S(k)) = 1 - P(x \in S(k)) = 1$$

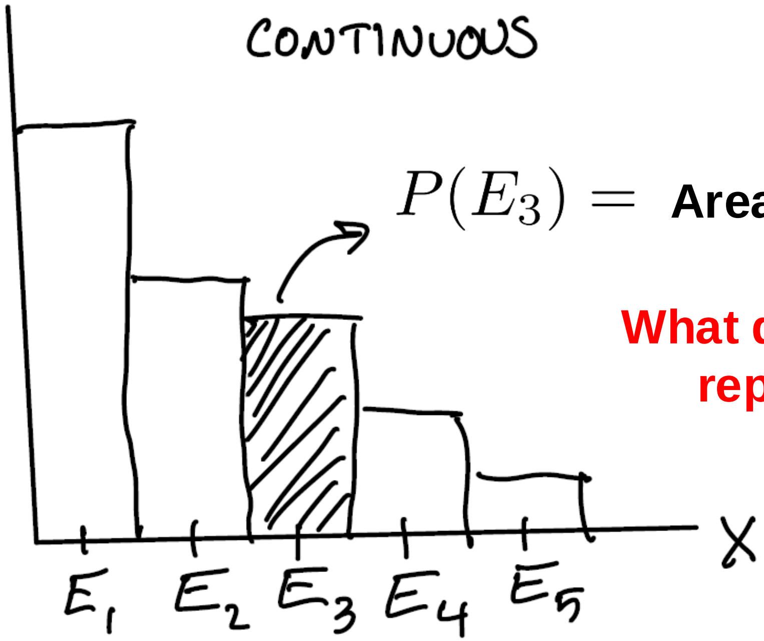
We seem to have a paradox!

Solution Rethink how we interpret probability in continuous setting

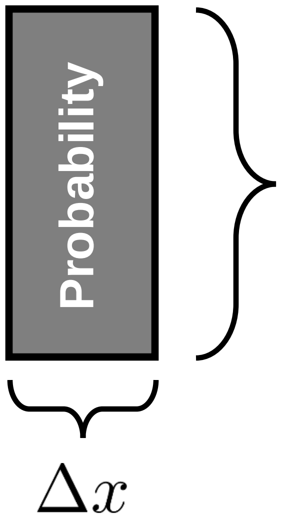
- Define events as *intervals* instead of discrete values
- Assign probability to those intervals

Continuous Probability





What does height represent?



Height $= \dfrac{\text{Probability}}{\Delta x}$

Height represents *probability per unit* in the x-direction

We call this a **probability density** (as opposed to probability mass)

Continuous Probability

- We denote the **probability density function** (PDF) as, $p(X)$
- An event E corresponds to an *interval* $a \leq X < b$
- The probability of an interval is given by the *area under the PDF,*

$$P(a \leq X < b) = \int_a^b p(X = x)\, dx$$

- Specific outcomes have zero probability $P(X = x) = P(x \leq X < x) = 0$
- But may have nonzero *probability density* $p(X = x)$

Continuous Probability Measures

Definition *The* *<u>cumulative distribution function</u>* *(CDF) of a real-valued continuous RV X is the function given by,*

$$P(x) = P(X \leq x)$$

Different ways to represent probability of interval, CDF is just a convention.

- Can easily measure probability of closed intervals,

$$P(a \leq X < b) = P(b) - P(a)$$

- If *X* is *absolutely continuous* (i.e. differentiable) then,

Fundamental Theorem of Calculus

$$p(x) = \frac{dP(x)}{dx} \quad \text{and} \quad P(t) = \int_{-\infty}^{t} p(x)\, dx$$

Where $p(x)$ is the *probability density function* (PDF)

Continuous Probability

Most definitions for discrete RVs hold, replacing PMF with PDF/CDF…

Two RVs X & Y are **independent** if and only if,

$$p(x,y) = p(x)p(y) \qquad \text{or} \qquad P(X \leq x, Y \leq y) = P(X \leq x)P(Y \leq y)$$

Conditionally independent given Z iff,

Shorthand: $P(x) = P(X \leq x)$

$$p(x,y \mid z) = p(x \mid z)p(y \mid z) \qquad \text{or} \qquad P(x,y \mid z) = P(x \mid z)P(y \mid z)$$

Probability chain rule,

$$p(x,y) = p(x)p(y \mid x) \qquad \text{and} \qquad P(x,y) = P(x)P(y \mid x)$$

Continuous Probability

…and by replacing summation with integration…

Law of Total Probability for continuous distributions,

$$p(x) = \int_{\mathcal{Y}} p(x, y) \, dy$$

Expectation of a continuous random variable,

$$\mathbf{E}[X] = \int_{\mathcal{X}} x \cdot p(x) \, dx$$

Covariance of two continuous random variables X & Y,

$$\mathbf{Cov}(X, Y) = \mathbf{E}[(X - \mathbf{E}[X])(Y - \mathbf{E}[Y])] = \int_{\mathcal{X}} \int_{\mathcal{Y}} (x - \mathbf{E}[X])(y - \mathbf{E}[Y]) p(x, y) \, dx dy$$

Continuous Probability

***Caution** Some technical subtleties arise in continuous spaces…*

For **discrete** RVs X & Y, the conditional

P(Y=y)=0 means impossible

$$P(X = x \mid Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)}$$

is **undefined** when P(Y=y) = 0 … no problem.

For **continuous** RVs we have,

$$P(X \leq x \mid Y = y) = \frac{P(X \leq x, Y = y)}{P(Y = y)}$$

but numerator and denominator are 0/0.

P(Y=y)=0 means improbable, but not impossible

Continuous Probability

Defining the conditional distribution as a limit fixes this…

$$P(X \leq x \mid Y = y) = \lim_{\delta \to 0} P(X \leq x \mid y \leq Y \leq y + \delta)$$

$$= \lim_{\delta \to 0} \frac{P(X \leq x, y \leq Y \leq y + \delta)}{P(y \leq Y \leq y + \delta)}$$

$$= \lim_{\delta \to 0} \frac{P(X \leq x, Y \leq y + \delta) - P(X \leq x, Y \leq y)}{P(Y \leq y + \delta) - P(Y \leq y)}$$

$$= \int_{-\infty}^{x} \lim_{\delta \to 0} \frac{\frac{\partial}{\partial x} P(u, y + \delta) - \frac{\partial}{\partial x} P(u, y)}{P(y + \delta) - P(y)} \, du$$

(Fundamental theorem of calculus)

(Assume interchange limit / integral)

$$= \int_{-\infty}^{x} \lim_{\delta \to 0} \frac{\left(\frac{\partial}{\partial x} P(u, y + \delta) - \frac{\partial}{\partial x} P(u, y)\right) / \delta}{\left(P(y + \delta) - P(y)\right) / \delta} \, du$$

(Multiply by $\frac{\delta}{\delta} = 1$)

$$= \int_{-\infty}^{x} \frac{\frac{\partial^2}{\partial x \partial y} P(u, y)}{\frac{\partial}{\partial y} P(y)} \, du \quad = \int_{-\infty}^{x} \frac{p(u, y)}{p(y)} \, du$$

(Definition of partial derivative)

(Definition PDF)

Definition The conditional PDF is given by,

$$p(x \mid y) = \frac{p(x, y)}{p(y)}$$

Useful Continuous Distributions

Uniform distribution on interval $[a, b]$,

$$p(x) = \begin{cases} 0 & \text{if } x \leq a, \\ \frac{1}{b-a} & \text{if } a \leq x \leq b, \\ 0 & \text{if } b \leq x \end{cases} \qquad P(X \leq x) = \begin{cases} 0 & \text{if } x \leq a, \\ \frac{x-a}{b-a} & \text{if } a \leq x \leq b, \\ 1 & \text{if } b \leq x \end{cases}$$

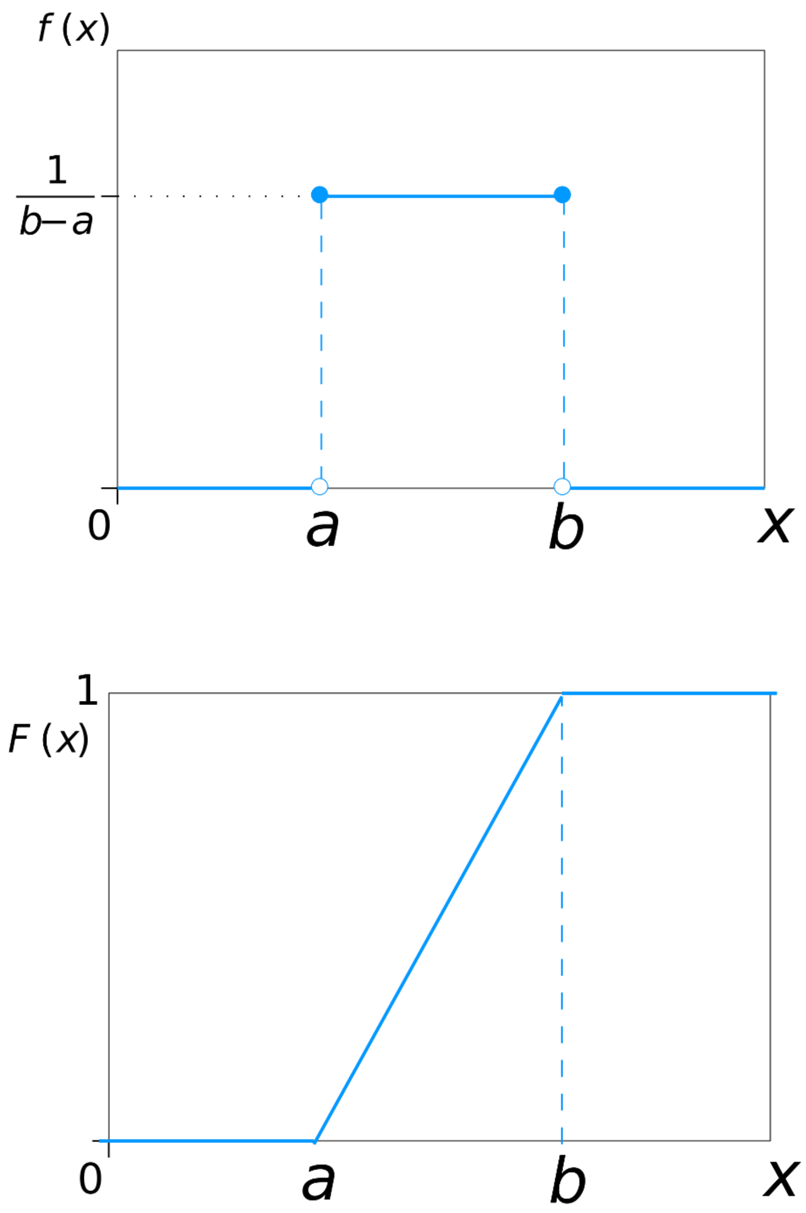
Say that $X \sim U(a, b)$ whose moments are,

$$\mathbf{E}[X] = \frac{b+a}{2} \qquad \mathbf{Var}[X] = \frac{(b-a)^2}{12}$$

Suppose $X \sim U(0, 1)$ and we are told $X \leq \frac{1}{2}$ what is the conditional distribution?

$$P(X \leq x \mid X \leq \tfrac{1}{2}) = U(0, \tfrac{1}{2})$$

Holds generally: Uniform closed under conditioning



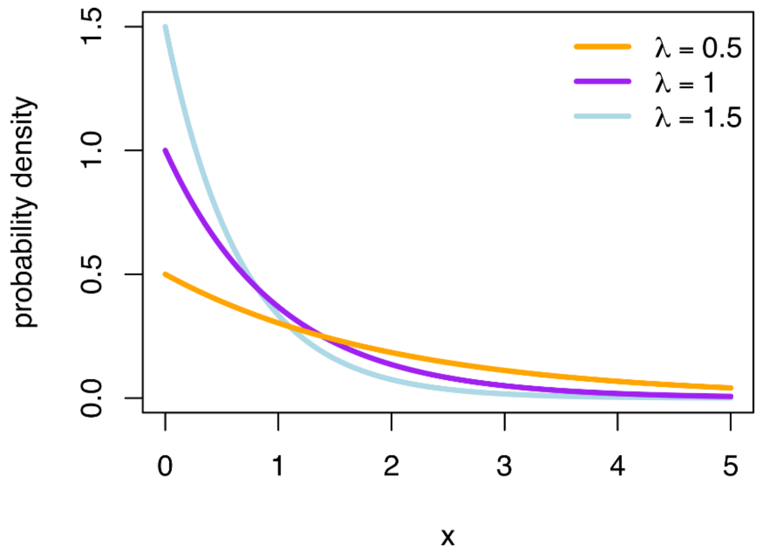
Useful Continuous Distributions

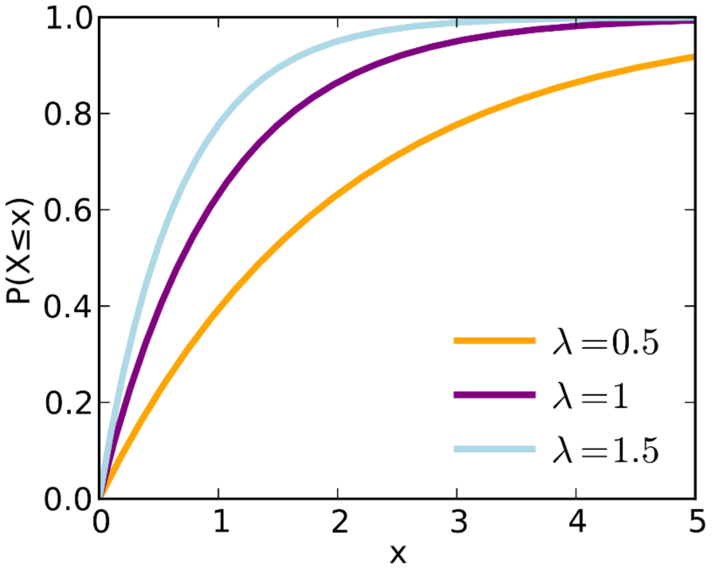
Exponential distribution with scale λ,

$$p(x) = \lambda e^{-\lambda x} \qquad P(x) = 1 - e^{-\lambda x}$$

for X>0. Moments given by,

$$\mathbf{E}[X] = \frac{1}{\lambda} \qquad \mathbf{Var}[X] = \frac{1}{\lambda^2}$$





Useful properties

- **Closed under conditioning** If $X \sim \text{Exponential}(\lambda)$ then,

$$P(X \geq s + t \mid X \geq s) = P(X \geq t) = e^{-\lambda t}$$

- **Minimum** Let $X_1, X_2, \ldots, X_N$ be i.i.d. exponentially distributed with scale parameters $\lambda_1, \lambda_2, \ldots, \lambda_N$ then,

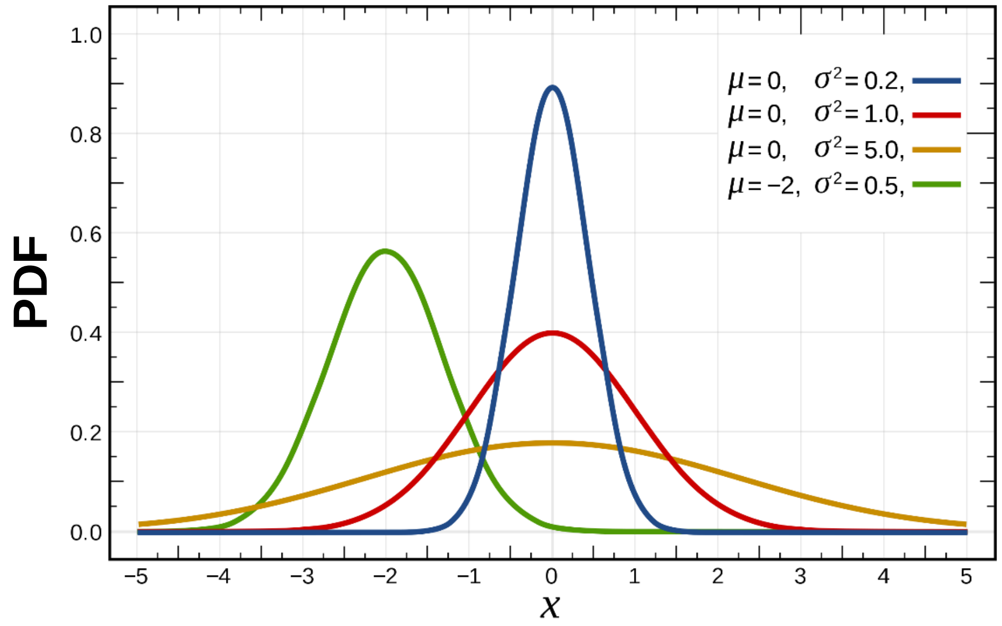
$$P(\min(X_1, X_2, \ldots, X_N)) = \text{Exponential}(\sum_i \lambda_i)$$

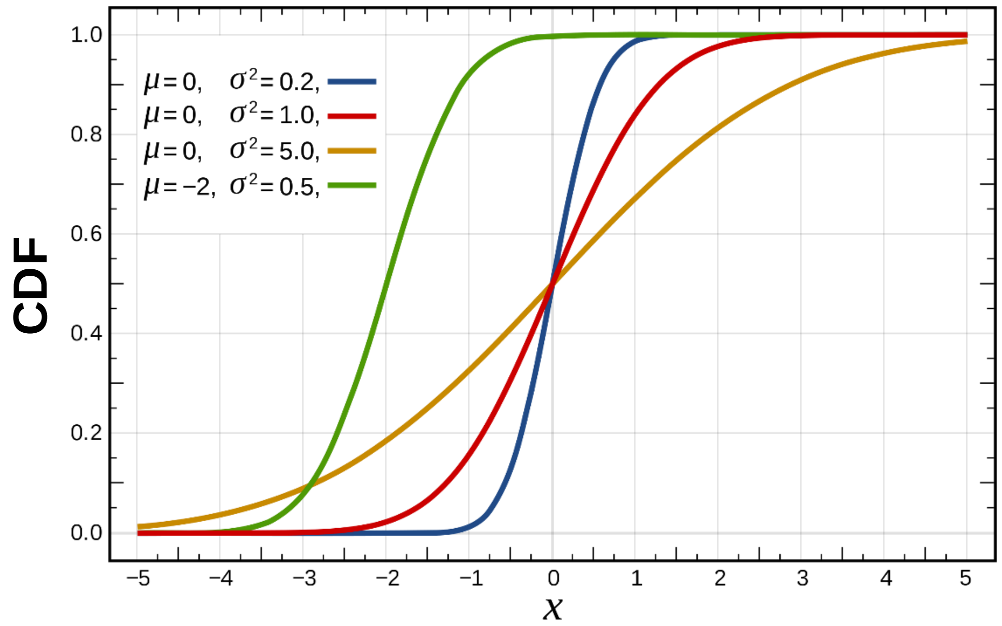
Useful Continuous Distributions

Gaussian (a.k.a. Normal) distribution with mean (location) μ and variance (scale) σ^2 parameters,

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp -\frac{1}{2}(x-\mu)^2/\sigma^2$$

We say $X \sim \mathcal{N}(\mu, \sigma^2)$.





Useful Properties

- Closed under additivity:

$$X \sim \mathcal{N}(\mu_x, \sigma_x^2) \qquad Y \sim \mathcal{N}(\mu_y, \sigma_y^2)$$

$$X + Y \sim \mathcal{N}(\mu_x + \mu_y, \sigma_x^2 + \sigma_y^2)$$

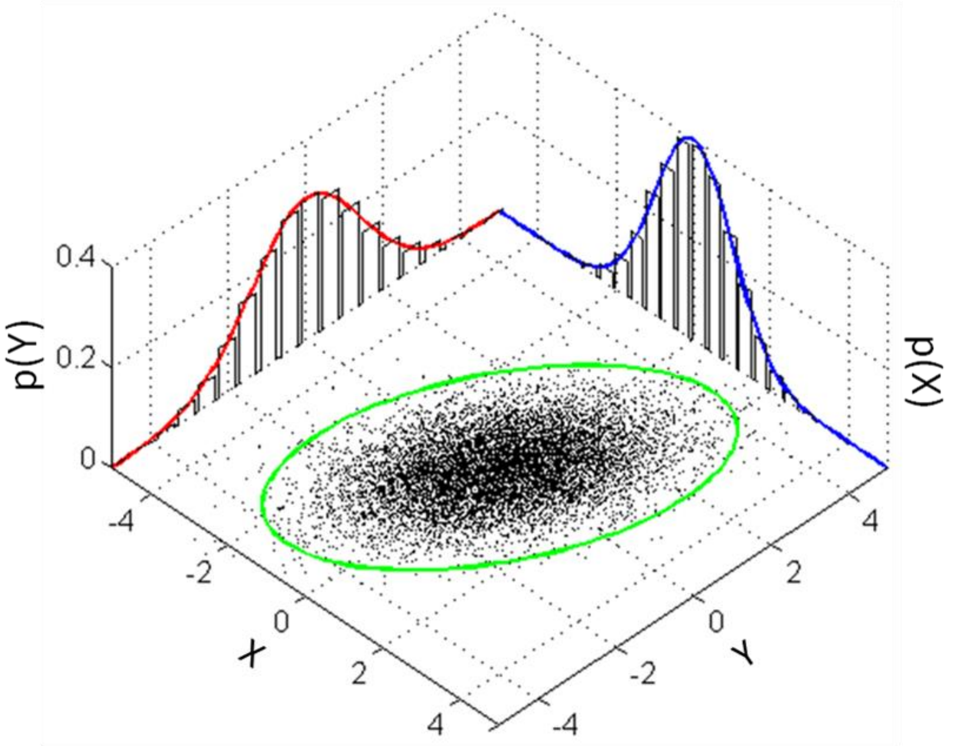
- Closed under linear functions (a and b constant):

$$aX + b \sim \mathcal{N}(a\mu_x + b, a^2\sigma_x^2)$$

Useful Continuous Distributions

Multivariate Gaussian On RV $X \in \mathcal{R}^d$ with mean $\mu \in \mathcal{R}^d$ and positive semidefinite covariance matrix $\Sigma \in \mathcal{R}^{d \times d}$,

$$p(x) = |2\pi\Sigma|^{-1/2} \exp -\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)$$



Moments given by parameters directly.

Useful Properties

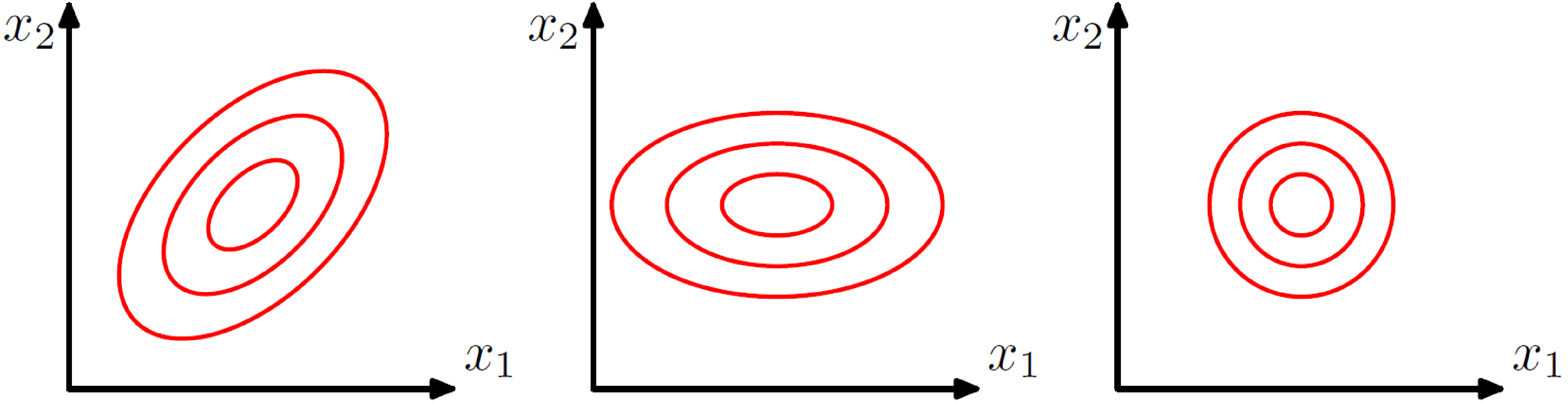
- Closed under additivity (same as univariate case)
- Closed under linear functions,

$$AX + b \sim \mathcal{N}(A\mu_x + b, A\Sigma A^T)$$

  Where $A \in \mathcal{R}^{m \times d}$ and $b \in \mathcal{R}^m$ (output dimensions may change)
- Closed under conditioning and marginalization

Covariance

Captures correlation between random variables…can be viewed as set of ellipses…



Positive Correlation

Uncorrelated

Uncorrelated and same variance (isotropic / spherical)

Covariance Matrix

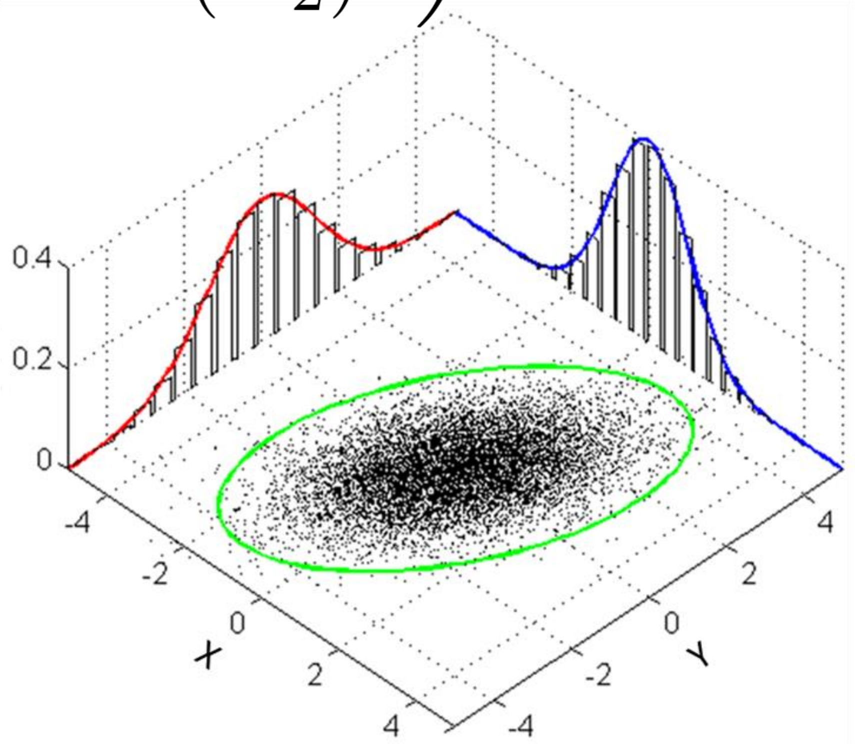
$$\Sigma = \mathrm{Cov}(X) = \begin{pmatrix} \mathrm{Var}(X_1) & \rho\sigma_{X_1}\sigma_{X_2} \\ \rho\sigma_{X_1}\sigma_{X_2} & \mathrm{Var}(X_2) \end{pmatrix}$$

Covariance Matrix

Marginal variance of just the RV X_1

$$\Sigma = \mathrm{Cov}(X) = \begin{pmatrix} \mathrm{Var}(X_1) & \rho\sigma_{X_1}\sigma_{X_2} \\ \rho\sigma_{X_1}\sigma_{X_2} & \mathrm{Var}(X_2) \end{pmatrix}$$

i.e. How "spread out" is the distribution in the X_1 dimension…



Covariance Matrix

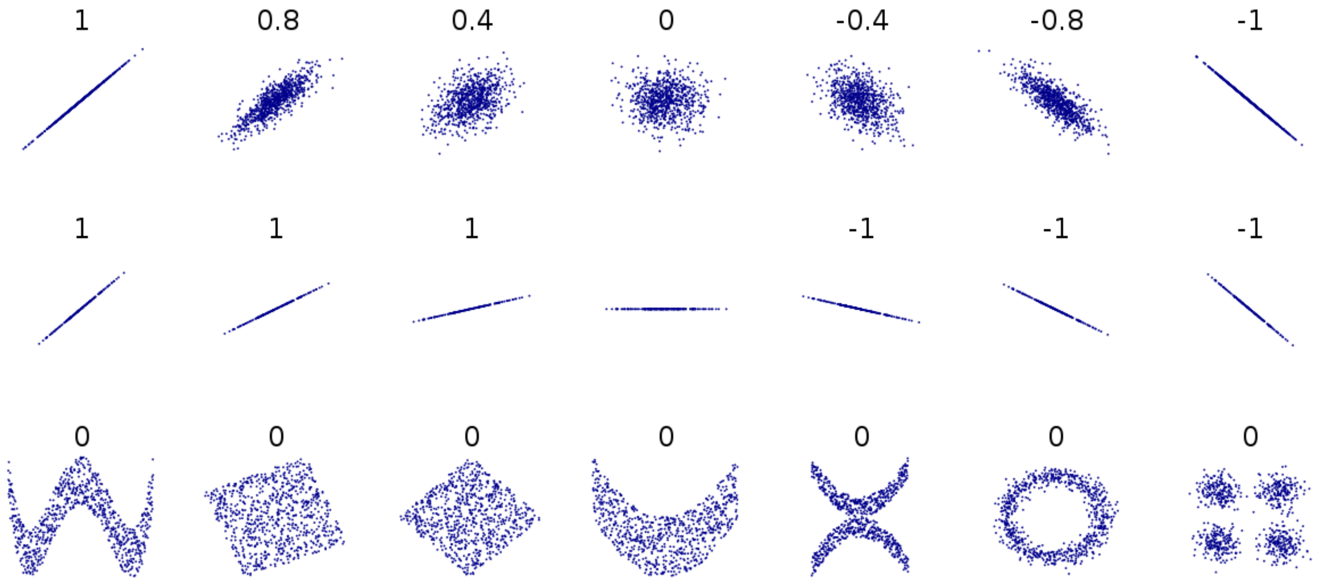
Correlation between X_1 and X_2

$$\Sigma = \mathrm{Cov}(X) = \begin{pmatrix} \mathrm{Var}(X_1) & \rho\sigma_{X_1}\sigma_{X_2} \\ \rho\sigma_{X_1}\sigma_{X_2} & \mathrm{Var}(X_2) \end{pmatrix}$$

Recall, correlation is given by:

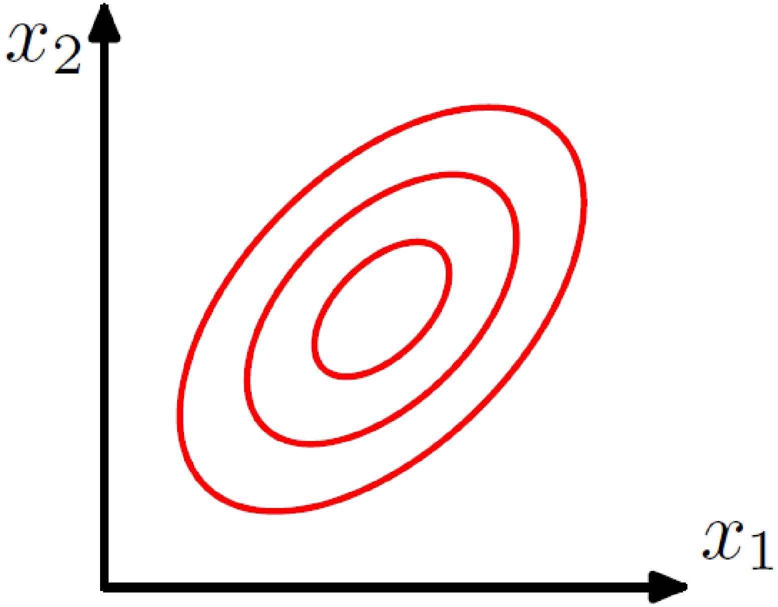
$$\rho = \frac{\mathbf{Cov}(X_1, X_2)}{\sigma_{X_1}\sigma_{X_2}}$$

Captures *linear dependence of RVs*



Covariance

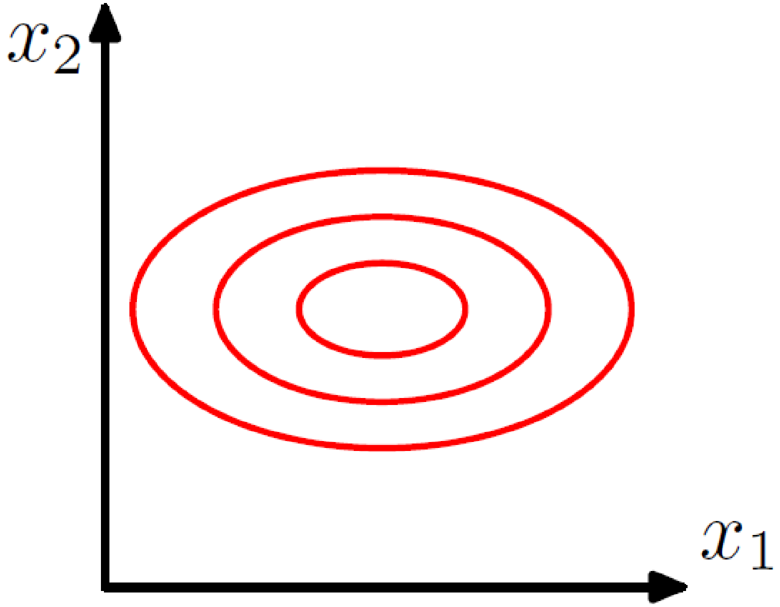
Captures correlation between random variables…can be viewed as set of ellipses…



Positive Correlation

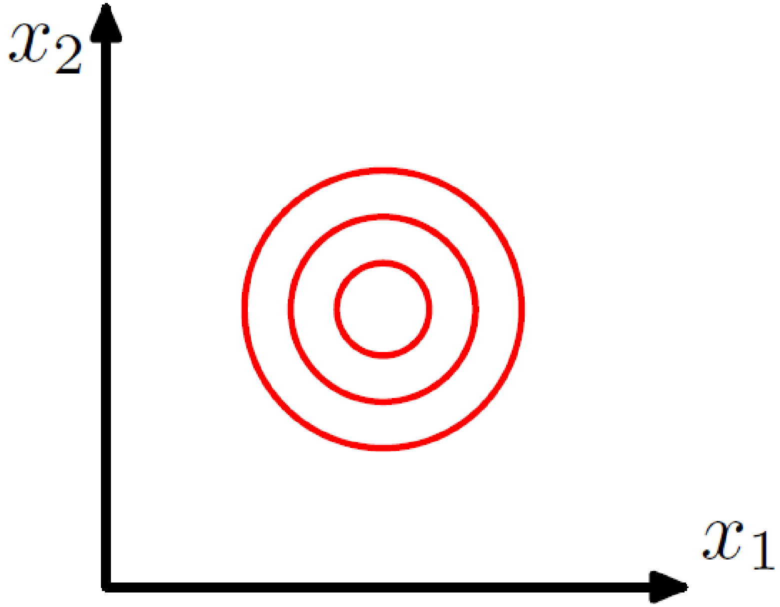
$\rho > 0$

Full matrix Σ



Uncorrelated

$$\Sigma = \begin{pmatrix} \sigma_{X_1}^2 & 0 \\ 0 & \sigma_{X_2}^2 \end{pmatrix}$$



Isotropic / Spherical

$$\Sigma = \begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix} = \sigma^2 I$$