Outline

- We did an introduction to estimation problems:
  - Point estimation
  - Biased vs unbiased estimator
  - Most efficient estimator among unbiased: Smallest variance
  - $\overline{X}$ is an unbiased estimator for the mean.

Today

- Point estimation of variance: Biased vs unbiased estimator
- Confidence intervals for the mean

Unbiased Estimator of Variance

The plugin estimator of variance $\frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2$ is biased.

Unbiased Estimator of Variance

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$Var(X) = E[(X-\mu)^2]$

plugin values from sample.

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We won't do a formal proof, instead we will do a simulation to verify this.

Simulation:

Do the following for different n (say $n = 2$, $n = 50$):

Generate 100000 random samples, each of size n, from Z.

Compute the value of estimator for each random sample.

Plot histogram of all values and show mean value of estimator.

Unbiased Estimator of Variance

The plugin estimator of variance $\frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2$ is biased.

$Z \rightarrow \frac{X-\mu}{\sigma}$ where X is normal.

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Standard normal distribution $\rightsquigarrow (0, 1)$ (mean, var)

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```
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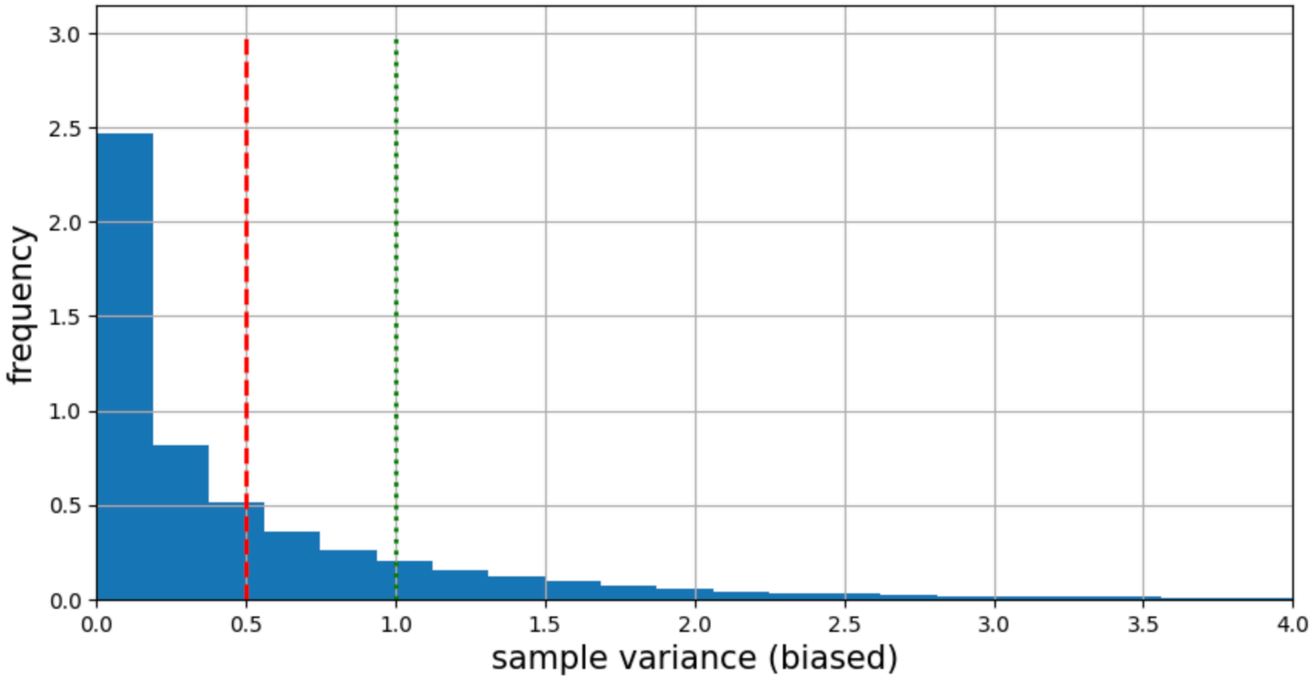

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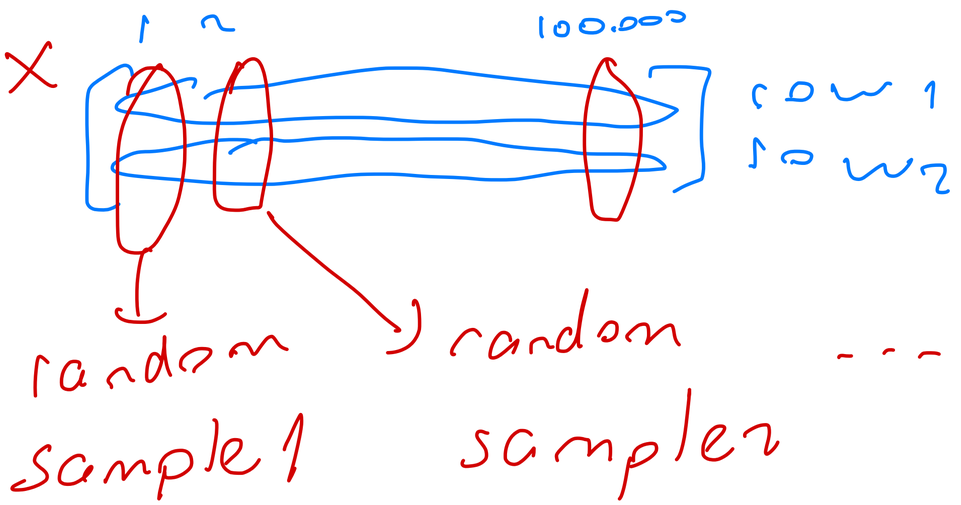
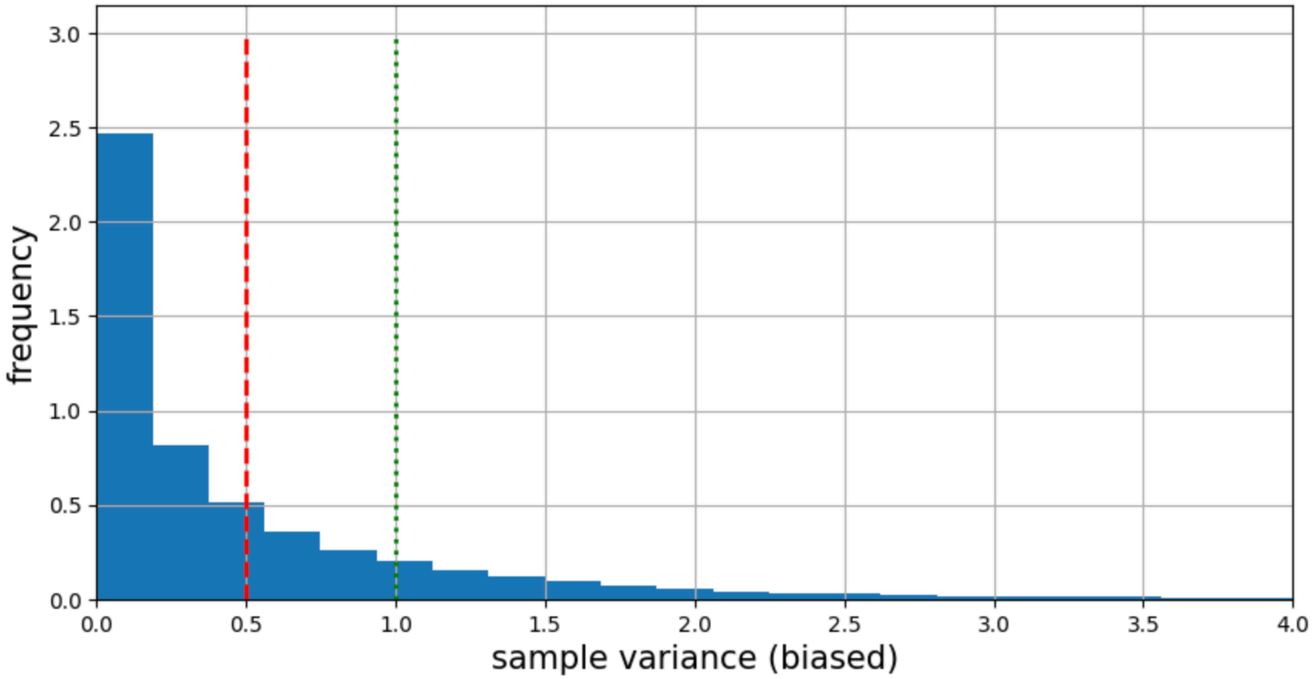
mean std. dev matrix





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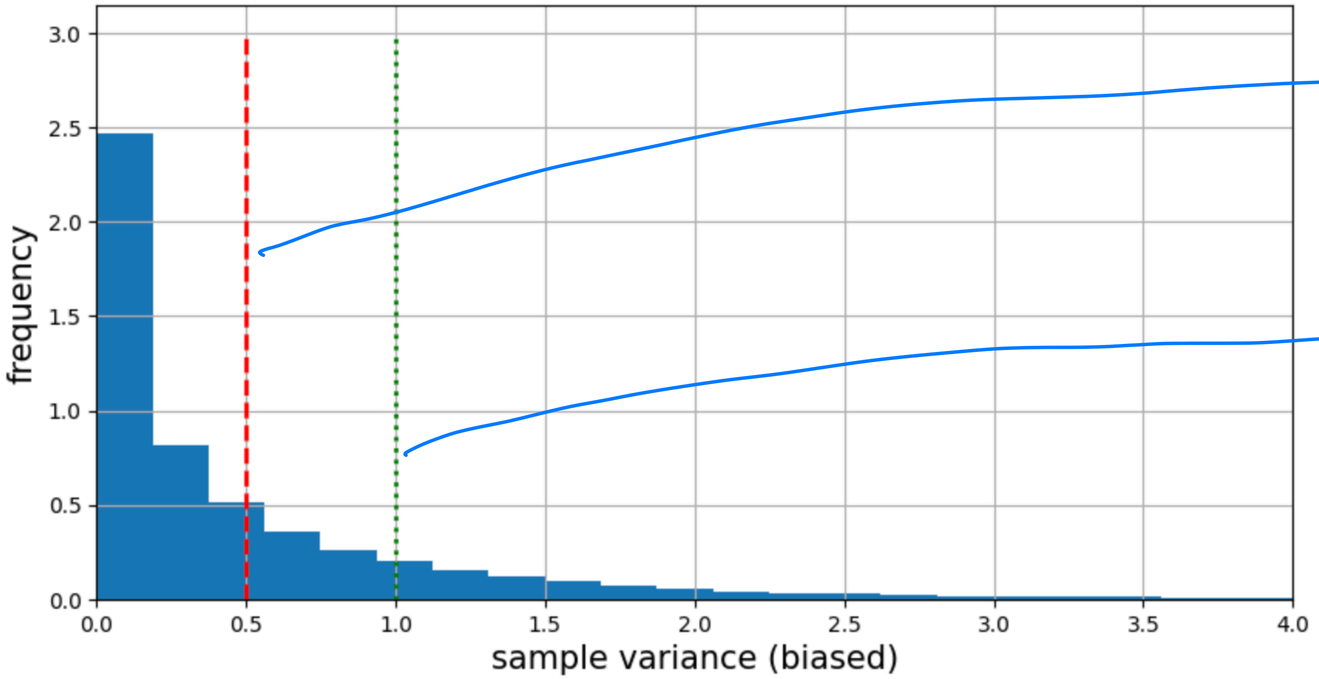

mean variance of the samples is 0.5

true value of the parameter (variance of the distribution) is 1.

Unbiased Estimator of Variance

The estimator of variance $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$ is unbiased.

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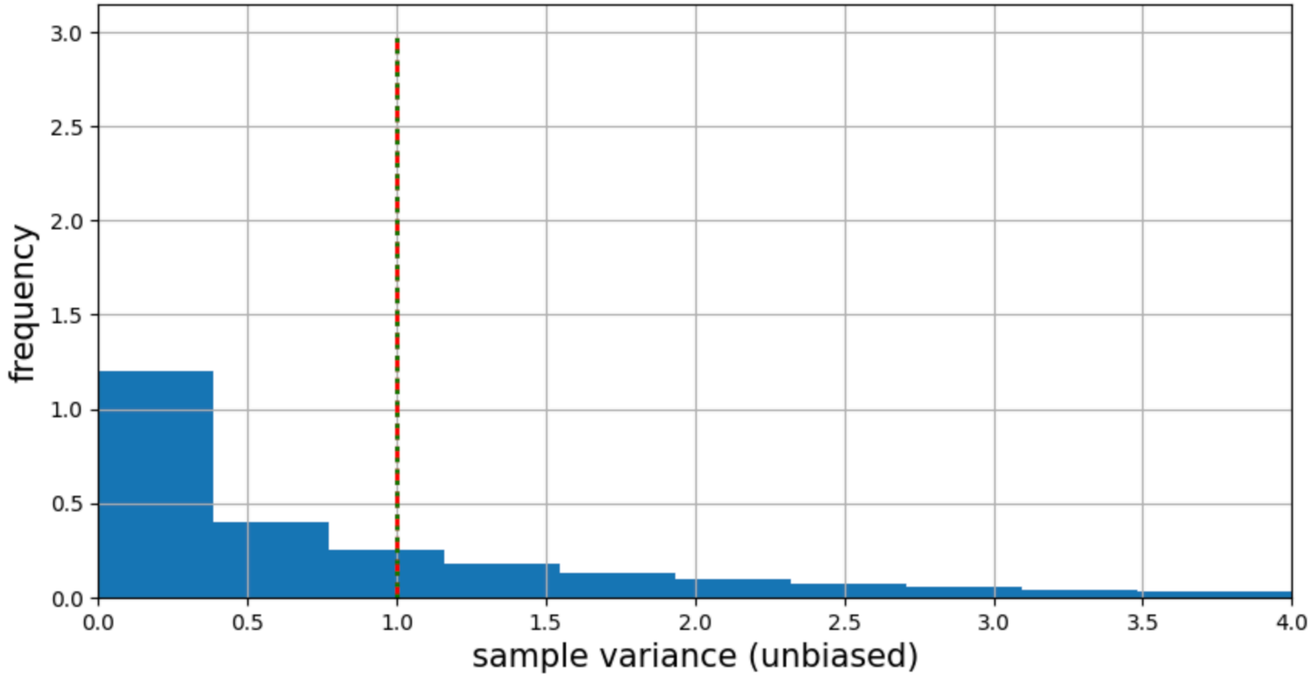

Formal proof in textbook.

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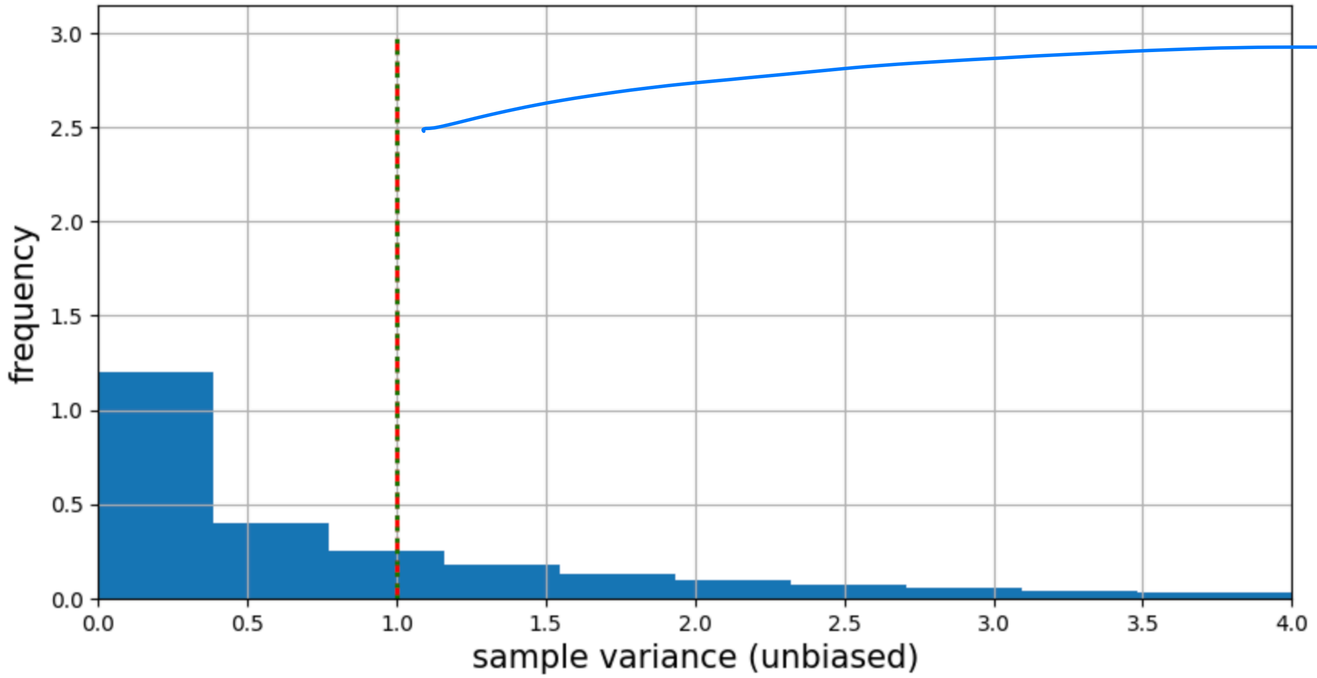

mean variance of samples is 1. ⇒ variance of the underlying distribution is also 1.

Unbiased Estimator of Variance

Dividing by n vs $n-1$ has a large effect when n is small.

For large n, they are almost the same!

```
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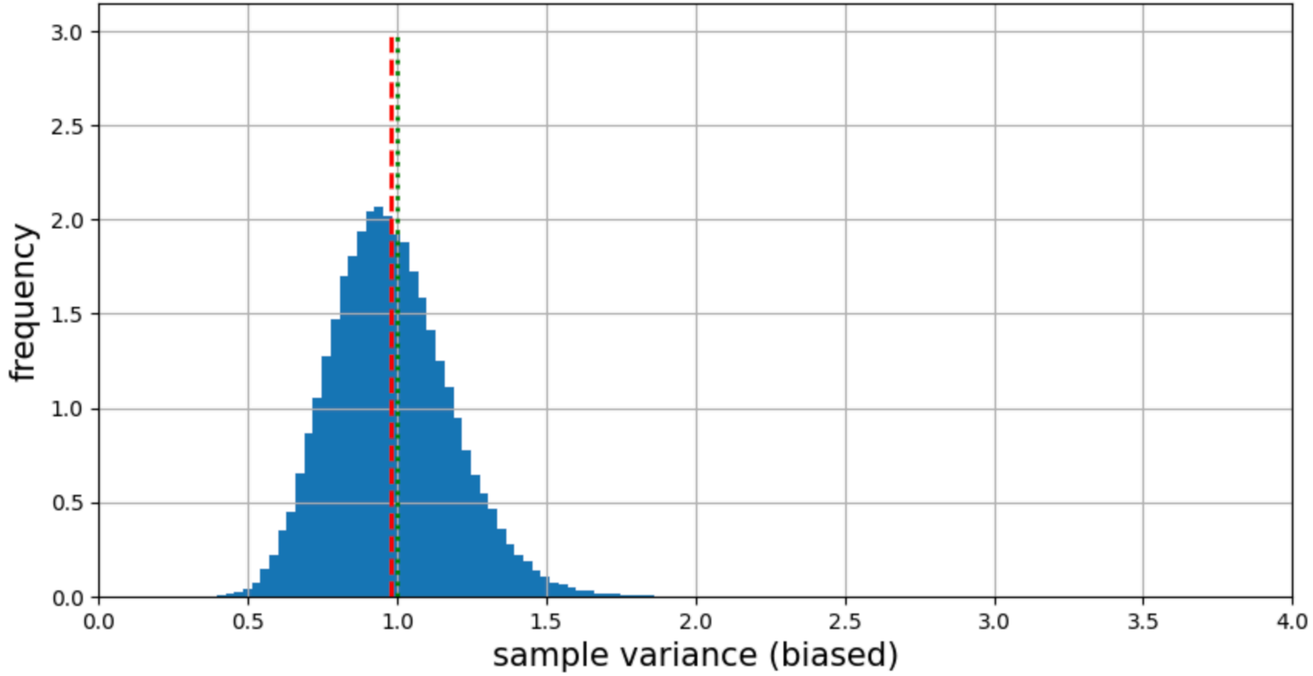

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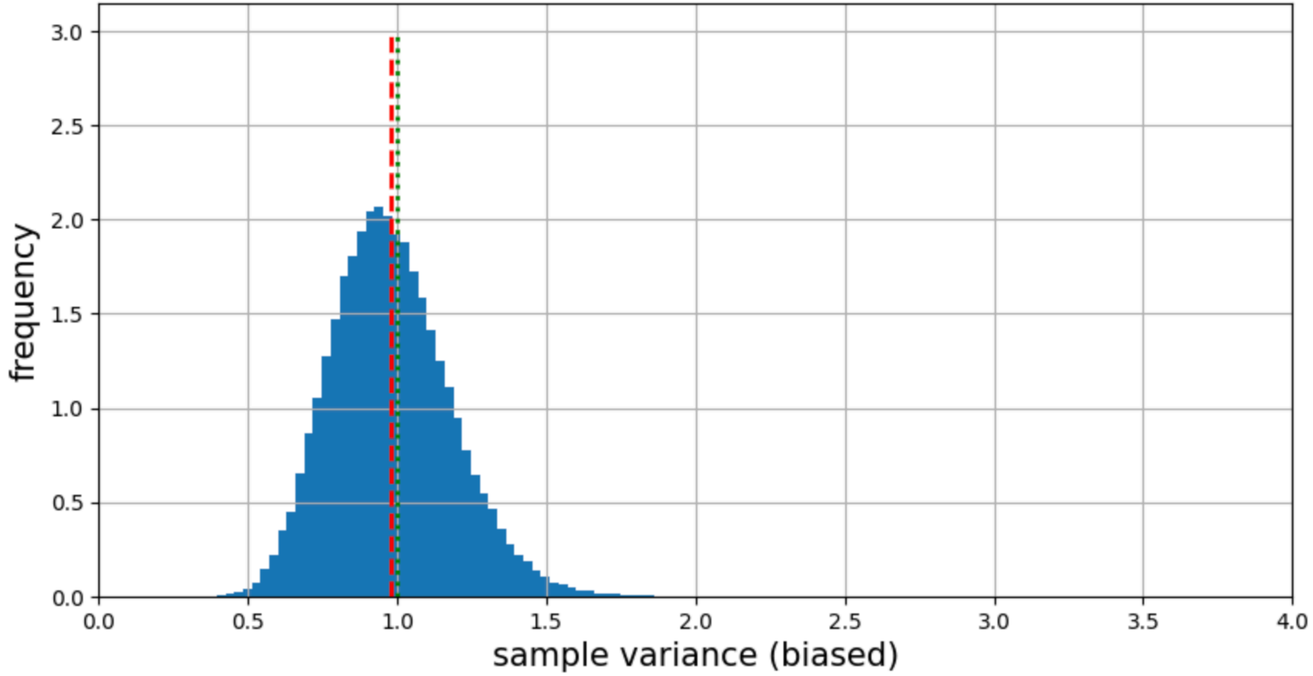
→ dividing by n
(the biased estimator)



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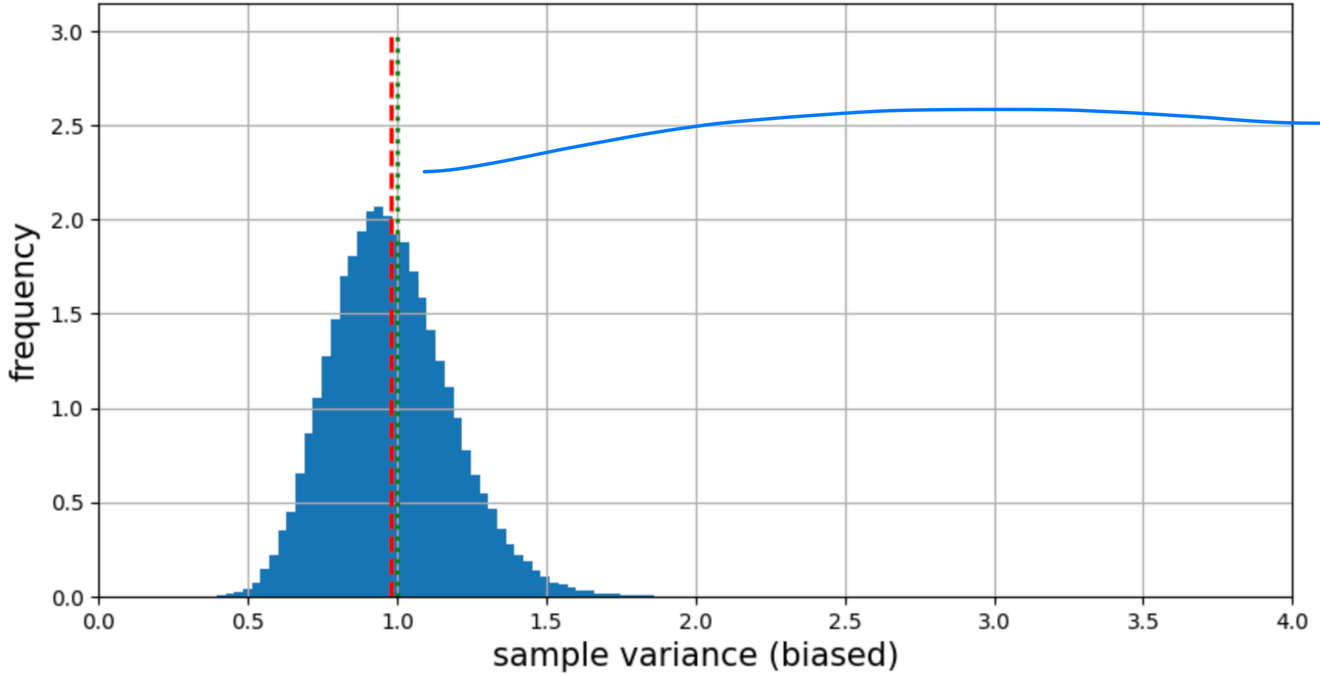
dividing by n (the biased estimator)



Since sample size (n) is large the mean of the estimator, even though it is biased, is almost same as variance of the distribution.

More Details About S^2

Observation from Previous Plots:

Distribution of S^2 is skewed for small dof and bell-curved for large dof.

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Not a coincidence!

Going back to Section 8.5 of the textbook:

Theorem 8.4: If S^2 is the variance of a random sample of size n taken from a normal population having the variance σ^2, then the statistic

$$\chi^2 = \frac{(n-1)S^2}{\sigma^2} = \sum_{i=1}^{n} \frac{(X_i - \bar{X})^2}{\sigma^2}$$

has a chi-squared distribution with $v = n - 1$ degrees of freedom.

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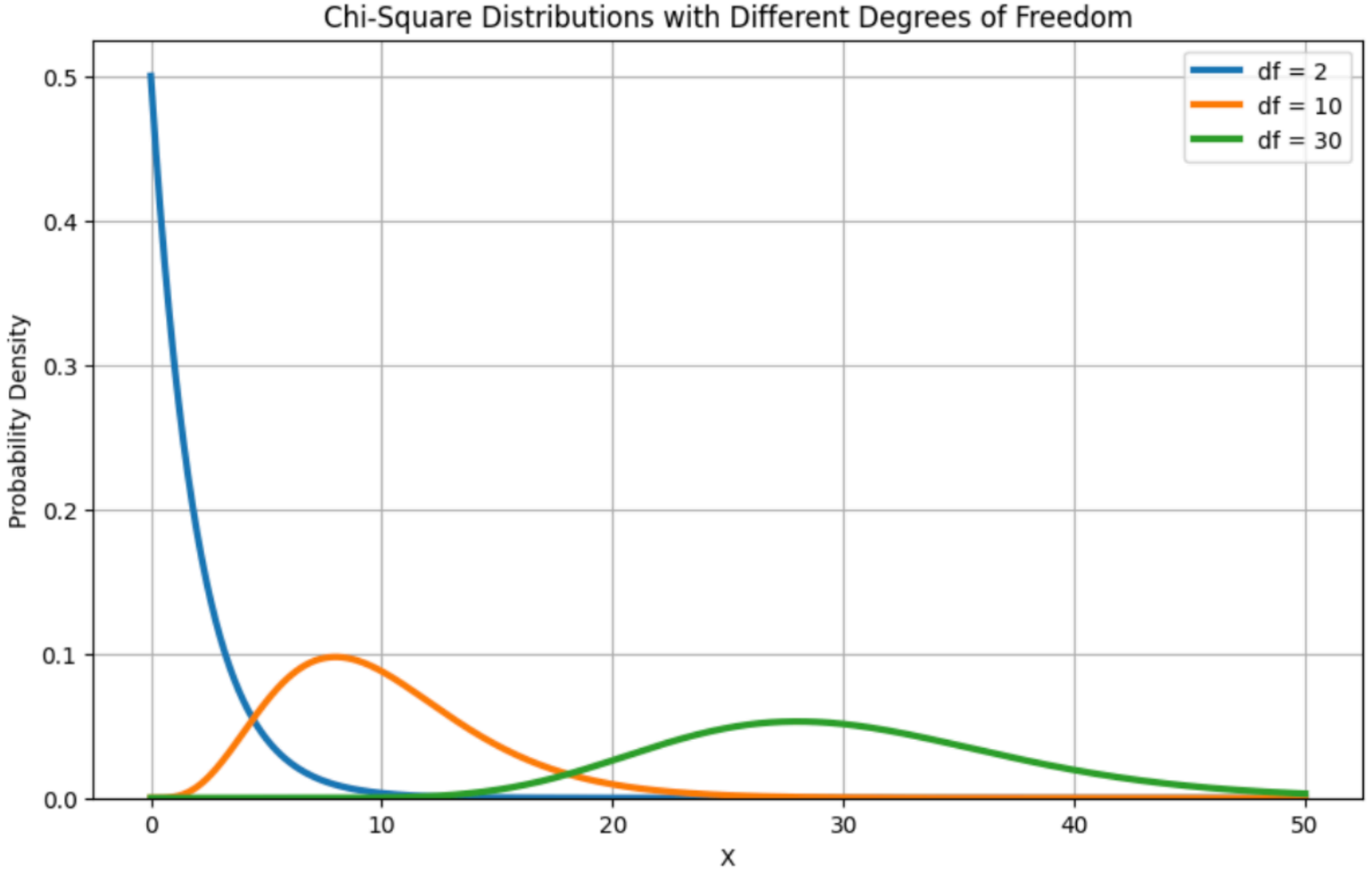
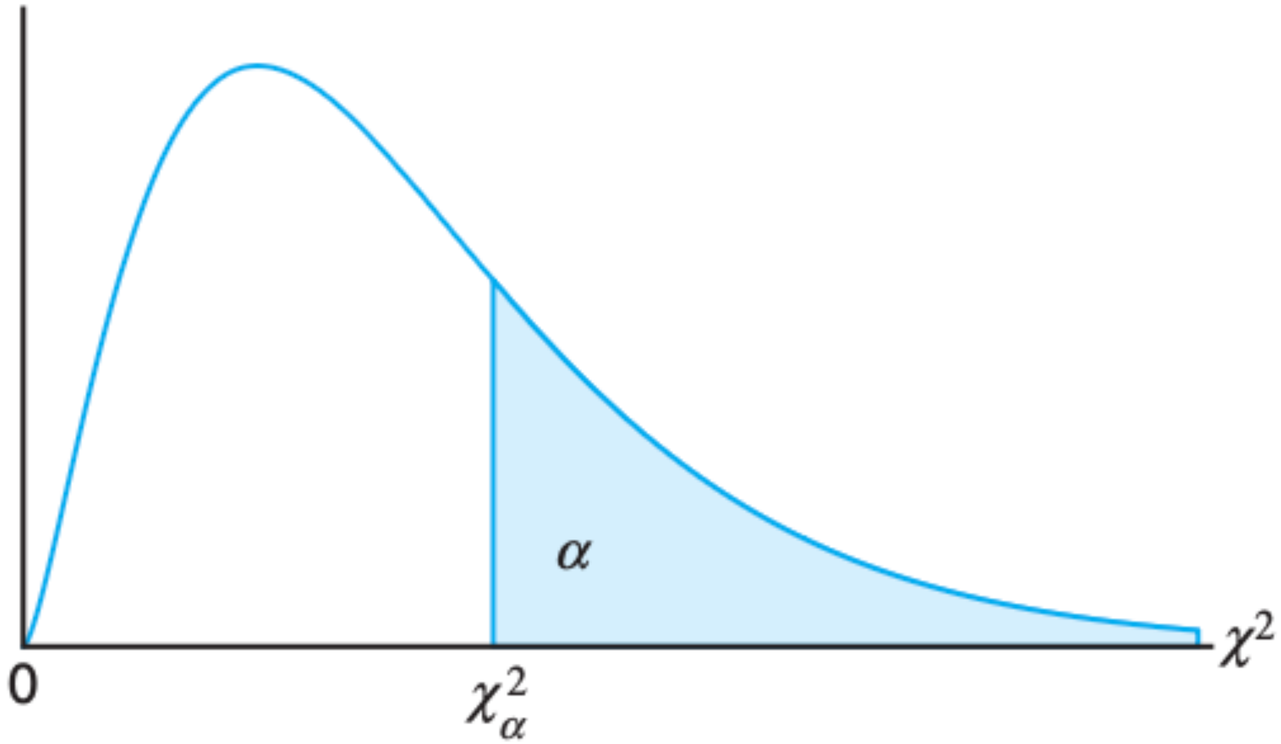
Not a coincidence!



χ^2

More About χ^2



χ^2 value above which the area is α

Note: Given α it can be found by *chi2.ppf*$(1-\alpha, \nu)$ using scipy.stats, where ν denotes dof.

More About χ^2

8.37 For a chi-squared distribution, find

(a) $\chi^2_{0.025}$ when $v = 15$;

Solution:

```
from scipy.stats import chi2
print(chi2.ppf(0.975, 15))
```

```
27.488392863442975
```

Confidence Intervals

We can apply them to any parameter but most common is mean μ.

We have seen: $\overline{X}$ is a sensible point estimator for mean μ.

Precise estimate, like $\overline{X} = 4$

but how much confidence do we have that $\mu = 4$?

Confidence interval:

Make statements like:

with $95\,\%$ *confidence* $\mu \in (\overline{X} - m, \overline{X} + m)$

Confidence Intervals

In general, for interval estimate of a parameter θ, if we find $\widehat{\Theta}_L$, $\widehat{\Theta}_U$ with:

$$P(\widehat{\Theta}_L < \theta < \widehat{\Theta}_U) = 1 - \alpha, \qquad \text{for } 0 < \alpha < 1$$

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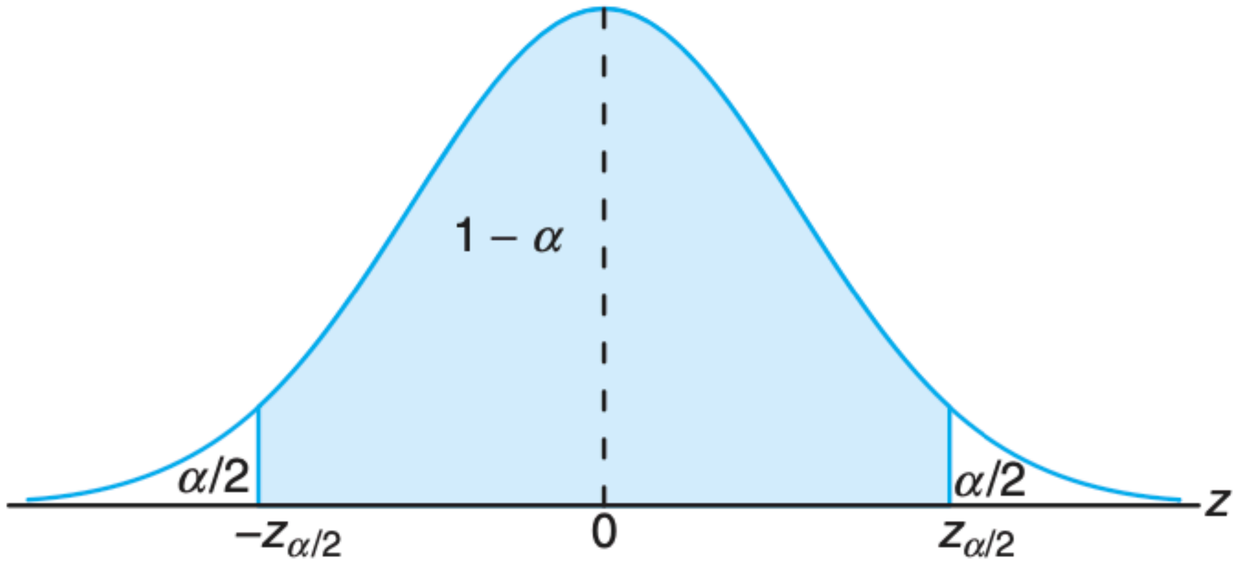
$\hat{\theta}_L < \theta < \hat{\theta}_U$ computed from the selected sample and is called a $100(1 - \alpha)\,\%$ confidence interval.

$\hat{\theta}_L$, $\hat{\theta}_U$ are called lower and upper confidence limits.

confidence coefficient (degree of confidence).

Confidence Interval for the Mean

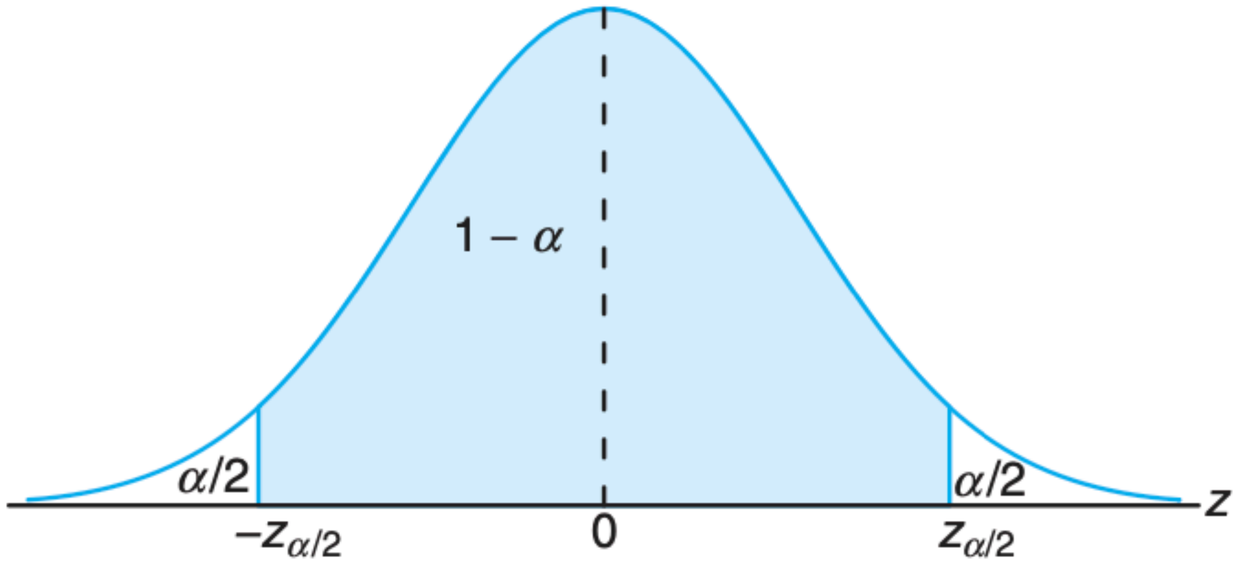


$$P(-z_{\alpha/2} < Z < z_{\alpha/2}) = 1 - \alpha$$

$\overline{X}$: mean of a large random sample.

With CLT we have $Z = \dfrac{\overline{X} - \mu}{\sigma/\sqrt{n}}$.

Confidence Interval for the Mean



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Doing some algebra we get:

Confidence Interval on μ, σ^2 Known

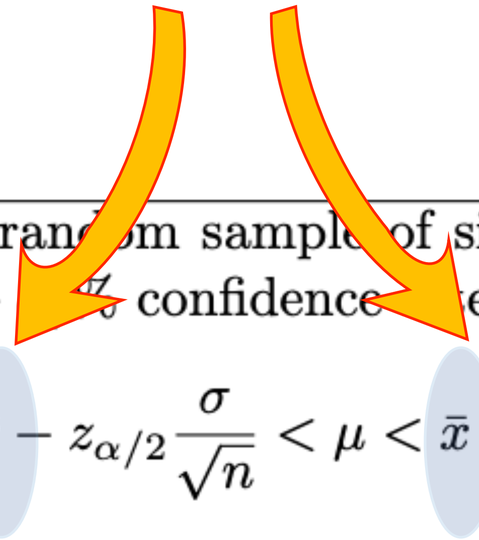
If $\bar{x}$ is the mean of a random sample of size n from a population with known variance σ^2, a $100(1-\alpha)\%$ confidence interval for μ is given by

$$\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}},$$

where $z_{\alpha/2}$ is the z-value leaving an area of $\alpha/2$ to the right.

Confidence Interval for the Mean

Different samples will yield different interval estimates.



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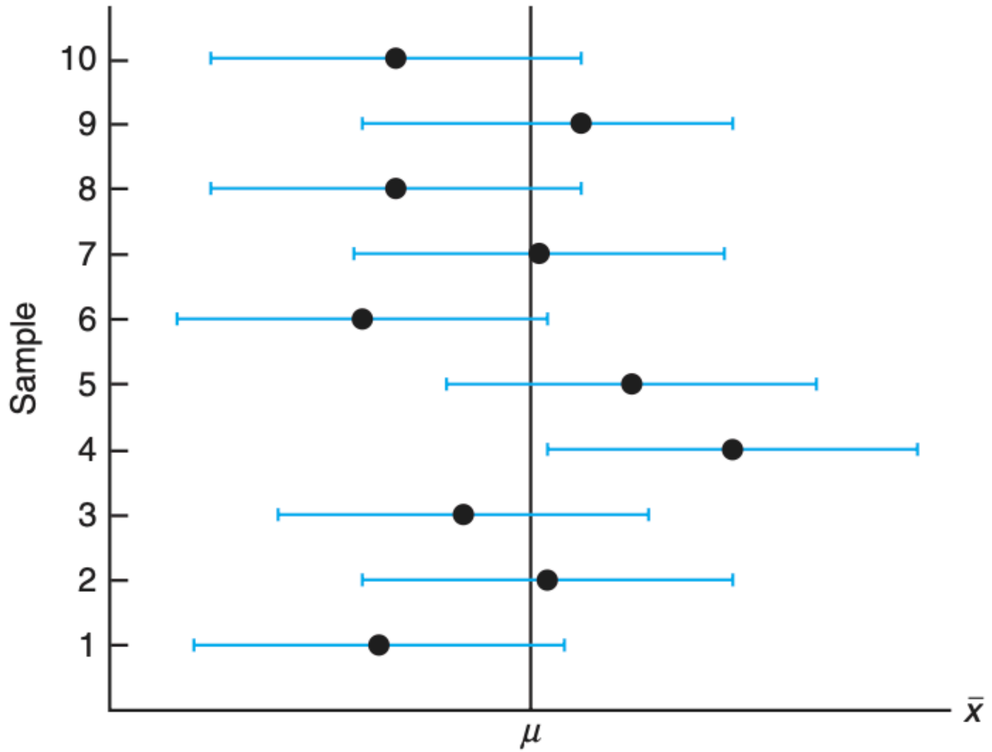

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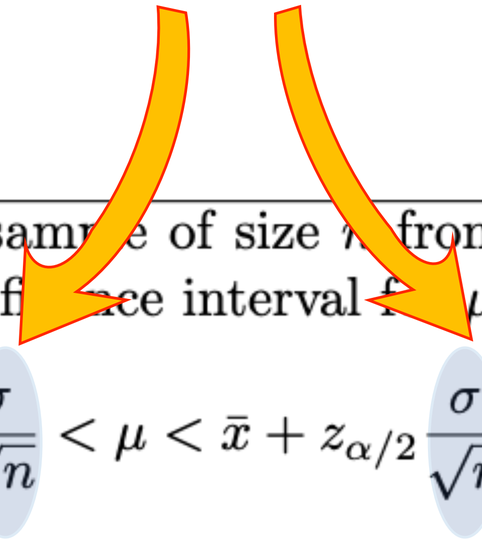
Confidence Interval for the Mean

(Standard deviation of $\overline{X}$) Standard error of $\overline{X}$



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Confidence Interval for the Mean

Confidence limit presented equivalently: $\bar{x} \pm z_{\alpha/2} s\,.\,e\,.\,(\bar{x})$

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Confidence Interval for the Mean

Example 9.2: The average zinc concentration recovered from a sample of measurements taken in 36 different locations in a river is found to be 2.6 grams per milliliter. Find the 95% and 99% confidence intervals for the mean zinc concentration in the river. Assume that the population standard deviation is 0.3 gram per milliliter.

Use $z_{0.025} = 1.96$ and $z_{0.005} = 2.575$

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95% confidence interval:

$$\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \Rightarrow 2.6 - 1.96\times\frac{0.3}{\sqrt{36}} < \mu < 2.6 + 1.96\times\frac{0.3}{\sqrt{36}}$$

$$\Rightarrow \underline{2.5 < \mu < 2.7}$$

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99% confidence interval:

$$\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \Rightarrow 2.6 - 2.575\times\frac{0.3}{\sqrt{36}} < \mu < 2.6 + 2.575\times\frac{0.3}{\sqrt{36}}$$

$$\Rightarrow \underline{2.47 < \mu < 2.73}$$

Confidence Interval for the Mean

Example: Number of citations of a random paper is a random variable with $\sigma = 5$. In a sample of 100 papers sample mean $\overline{X}$ is 7.5. What is the $95\,\%$ confidence interval for mean μ?

Use $z_{0.025} = 1.96$ or $z_{0.05} = 1.645$

Choose which one to use.

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95 % confidence interval =

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$$\Rightarrow 7.5 - 0.98 < \mu < 7.5 + 0.98$$

$$\Rightarrow \underline{6.52 < \mu < 8.48}$$