Outline

Review

- Hypothesis Testing
  - One-sided vs two-sided
  - Critical region/value, Significance level α, and Type I error

Today

- Type II error β
- Relationship between α, β, sample size
- Example with continuous random variable
- Presenting conclusions with P-value

Type II Error

Type I error is not the only type of error we can have in hypothesis testing.

We can also have Type II error.

Nonrejection of the null hypothesis when it is false is called a **type II error**.

β: Probability of making Type II error in the test.

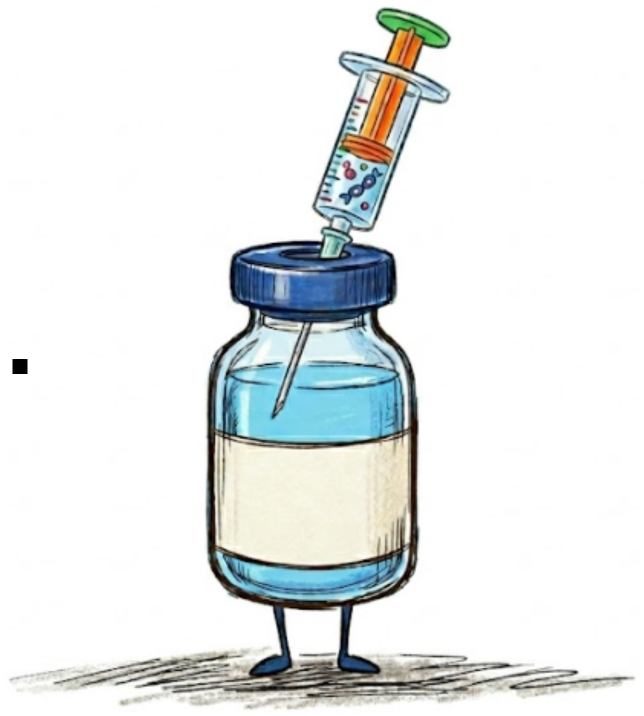
Hypothesis Testing

Table 10.1: Possible Situations for Testing a Statistical Hypothesis

	H_0 is true	**H_0 is false**
Do not reject H_0	Correct decision	Type II error
Reject H_0	Type I error	Correct decision

β and Type II Error

Example: One-sided test with a discrete random variable



Testing whether a new vaccine has effectiveness more than 25%.

$$H_0: \ p \leq 0.25 \qquad H_1: \ p > 0.25$$

Let's see β and Type II error. We need specific alternative hypothesis p.

H_0 is not rejected even though $p = 0.5$

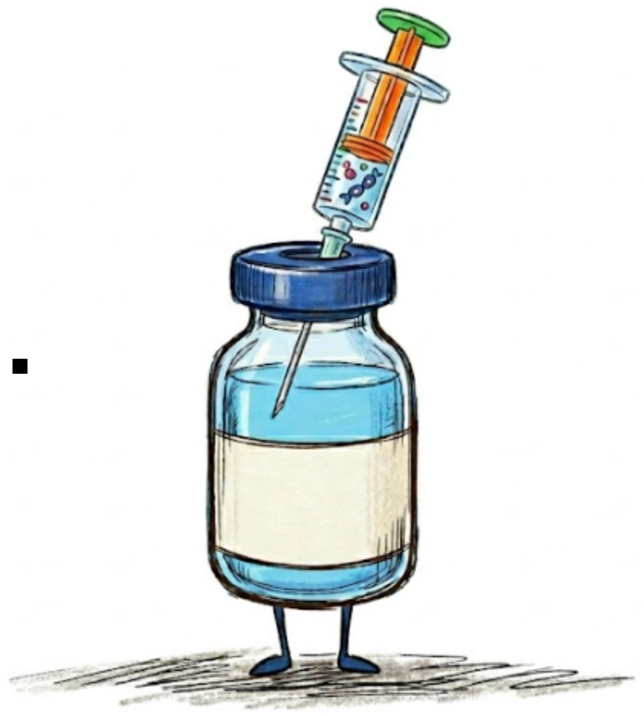
$$\beta = P(\text{type II error}) = P\left(X \leq 8 \text{ when } p = \frac{1}{2}\right)$$

$$= \sum_{x=0}^{8} b\left(x; 20, \frac{1}{2}\right) = 0.2517.$$

β and Type II Error

Example: One-sided test with a discrete random variable

Testing whether a new vaccine has effectiveness more than 25%.



$$H_0: \ p \le 0.25 \qquad H_1: \ p > 0.25$$

Let's see β and Type II error. We need specific alternative hypothesis p.

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$$= \sum_{x=0}^{8} b\left(x; 20, \frac{1}{2}\right) = 0.2517.$$

H_0 is not rejected even though $p = 0.7$

$$\beta = P(\text{type II error}) = P(X \le 8 \text{ when } p = 0.7)$$

$$= \sum_{x=0}^{8} b(x; 20, 0.7) = 0.0051.$$

β and Type II Error

H_0 is not rejected even though $p = 0.5$

$$\beta = P(\text{type II error}) = P\left(X \leq 8 \text{ when } p = \frac{1}{2}\right)$$

$$= \sum_{x=0}^{8} b\left(x; 20, \frac{1}{2}\right) = 0.2517.$$

Maybe this is large for our purposes.

Relationship between α, β, and Sample Size

β can be decreased by:

Increasing size of critical region, but this increases α.

H_0 is not rejected even though $p = 0.5$

$$\beta = P(\text{type II error}) = P\left(X \leq 8 \text{ when } p = \frac{1}{2}\right)$$

$$= \sum_{x=0}^{8} b\left(x; 20, \frac{1}{2}\right) = 0.2517.$$

Relationship between α, β, and Sample Size

β can be decreased by:

Increasing size of critical region, but this increases α.

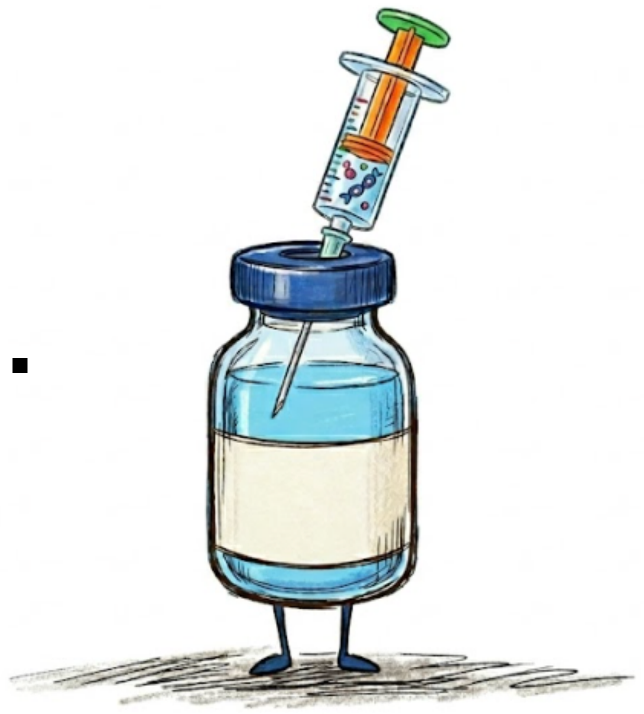
Increasing size of sample (in fact it decreases α as well).

H_0 is not rejected even though $p = 0.5$

$$\beta = P(\text{type II error}) = P\left(X \leq 8 \text{ when } p = \frac{1}{2}\right)$$

$$= \sum_{x=0}^{8} b\left(x; 20, \frac{1}{2}\right) = 0.2517.$$

α Decreases as Sample Size Increases



Example: One-sided test with a discrete random variable

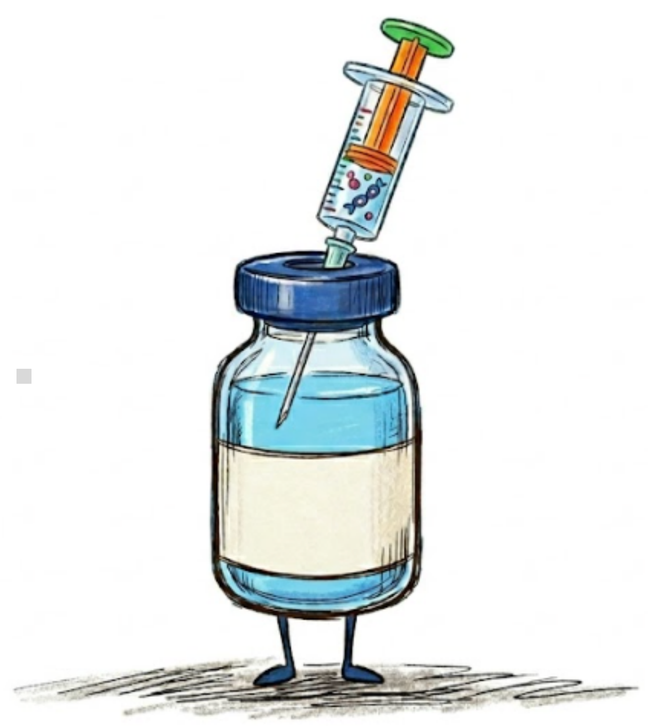
Testing whether a new vaccine has effectiveness more than $25\ \%$.

$$H_0 :\ p \leq 0.25 \qquad H_1 :\ p > 0.25$$

α Decreases as Sample Size Increases

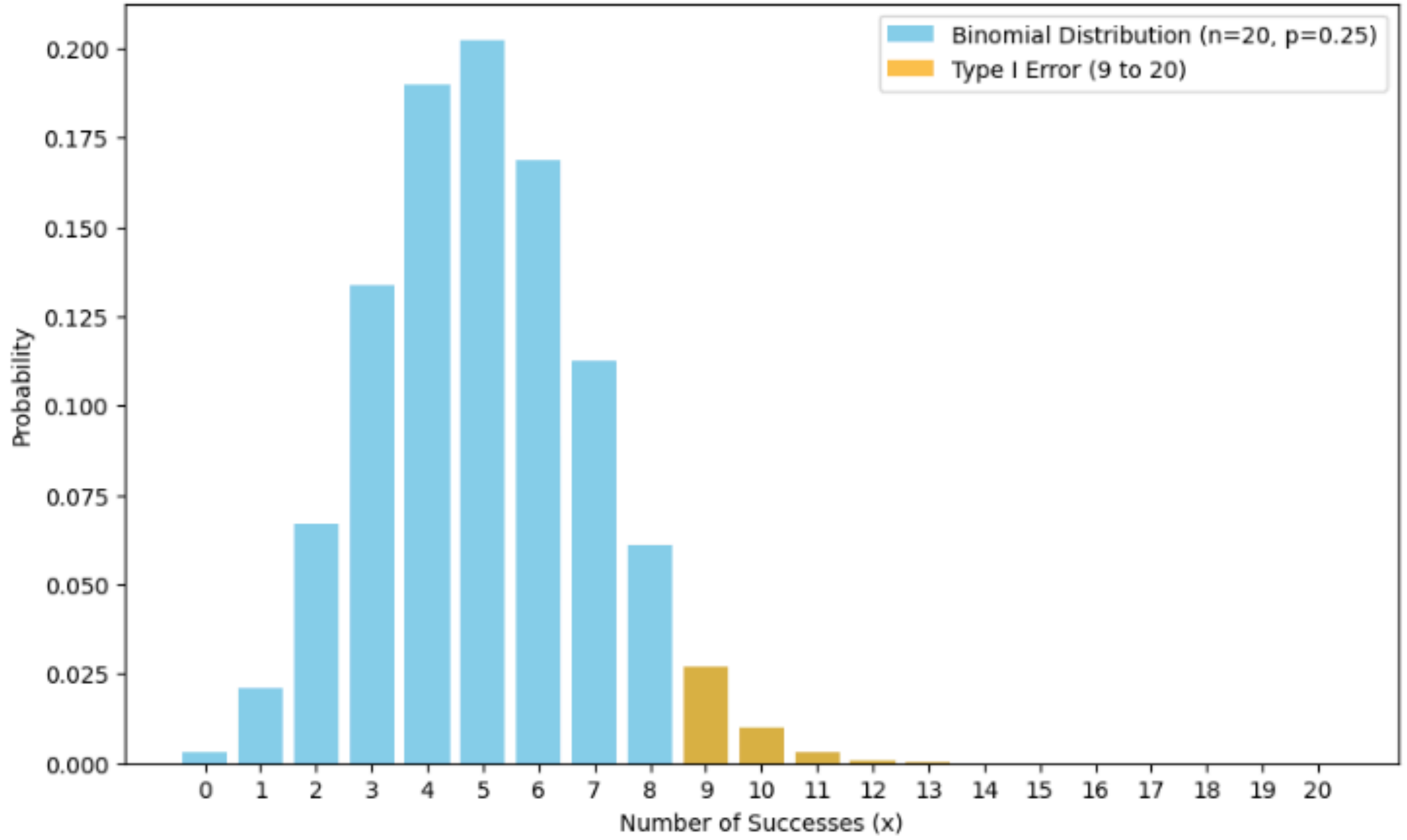
Example: One-sided test with a discrete random variable

Testing whether a new vaccine has effectiveness more than $25\ \%$.



$$H_0:\ p \le 0.25 \qquad H_1:\ p > 0.25$$

Probability of Type I Error (Sample size 20)



Assuming H_0 is True.

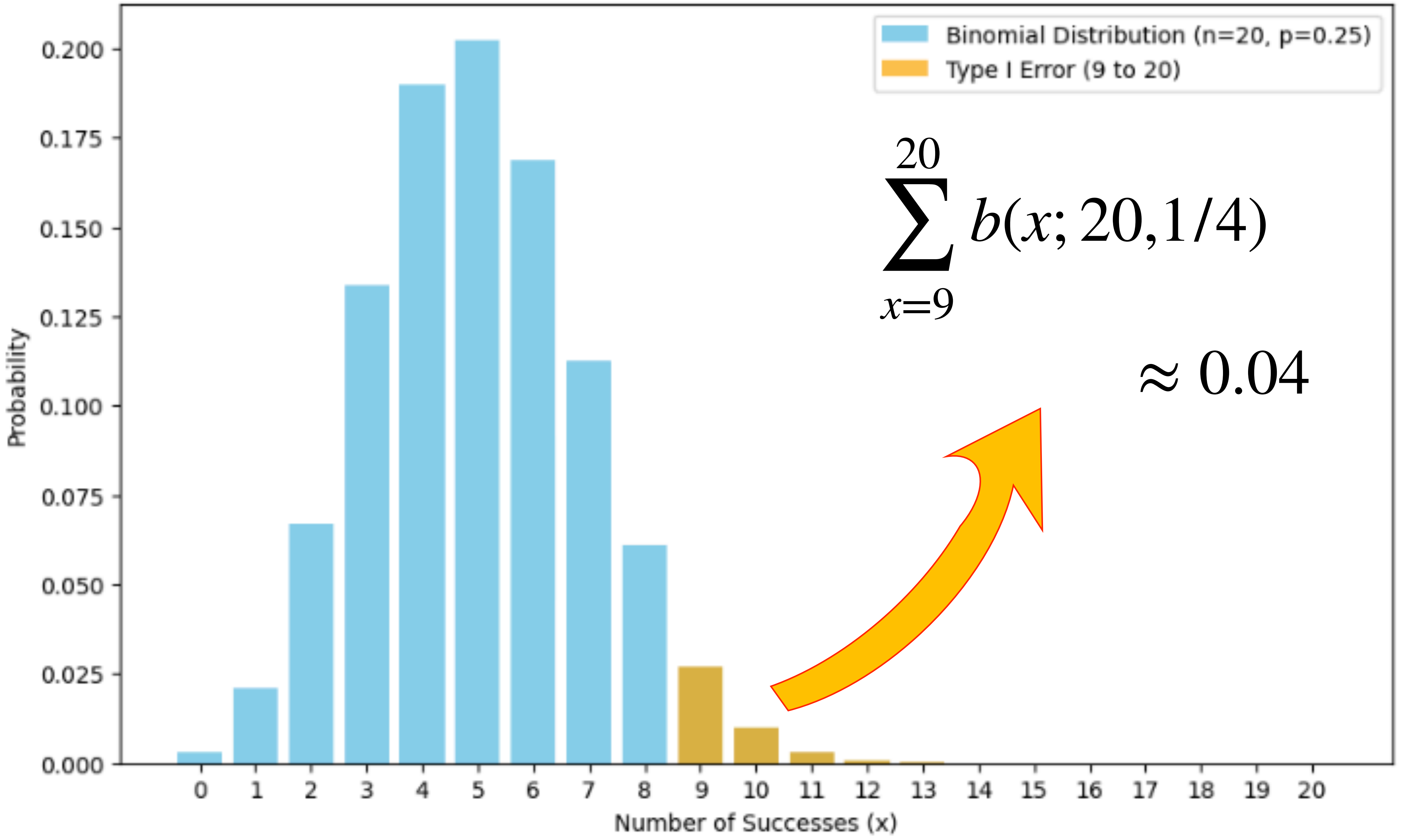
α Decreases as Sample Size Increases

Example: One-sided test with a discrete random variable

Testing whether a new vaccine has effectiveness more than $25\ \%$.

$$H_0 : \ p \leq 0.25 \qquad H_1 : \ p > 0.25$$

Probability of Type I Error (Sample size 20)



$$\sum_{x=9}^{20} b(x; 20, 1/4)$$

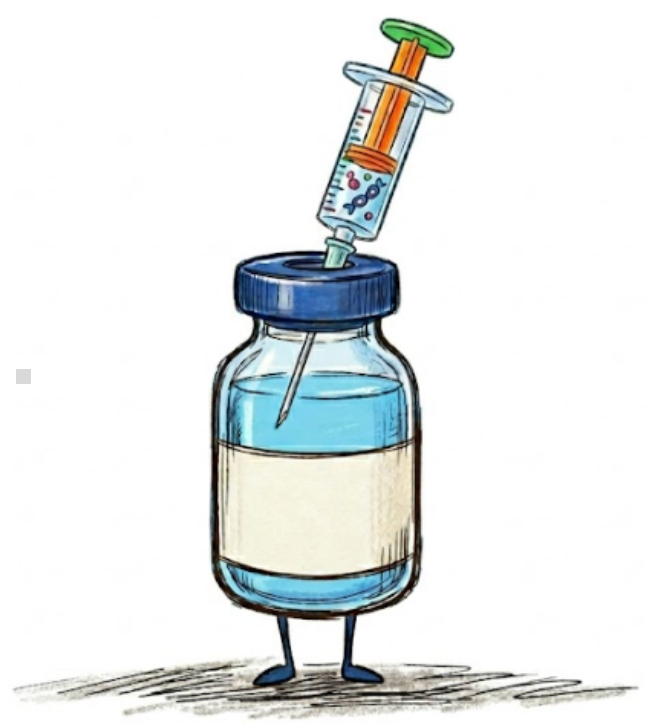
$$\approx 0.04$$

Assuming H_0 is True.

α Decreases as Sample Size Increases

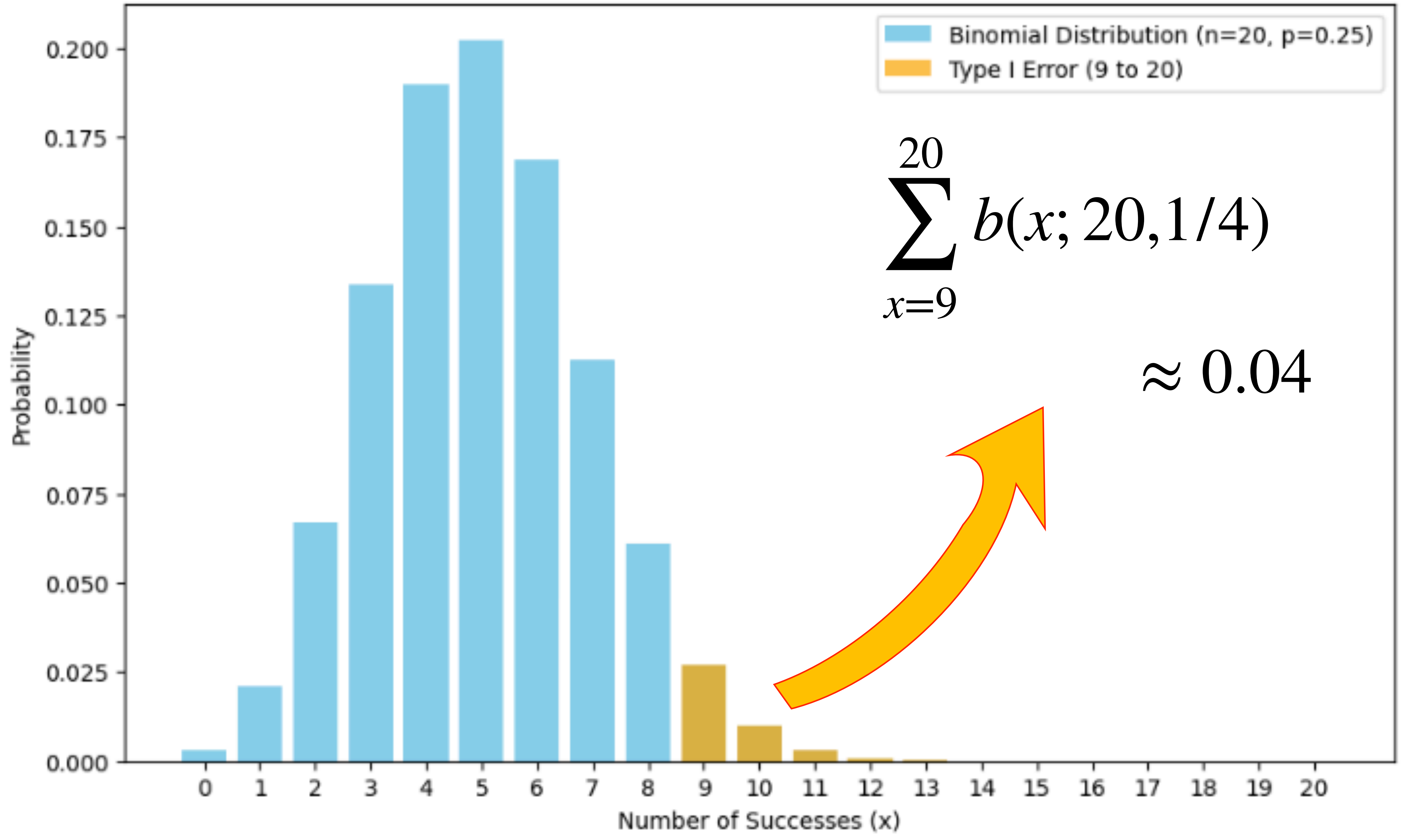
Example: One-sided test with a discrete random variable

Testing whether a new vaccine has effectiveness more than $25\ \%$.



$$H_0:\ p \le 0.25 \qquad H_1:\ p > 0.25$$

Probability of Type I Error (Sample size 20)

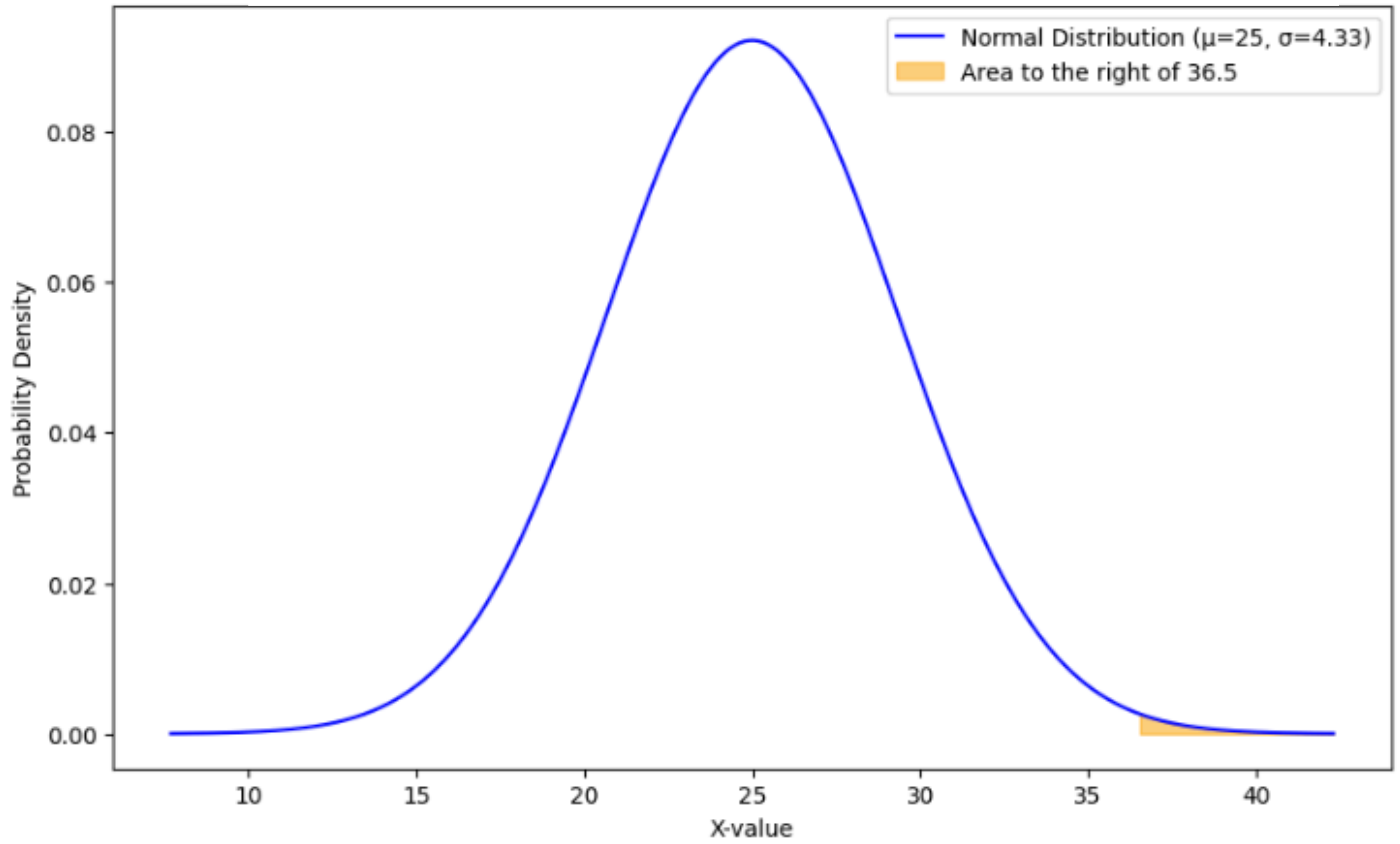


$$\sum_{x=9}^{20} b(x; 20, 1/4)$$

$$\approx 0.04$$

Assuming H_0 is True.

Probability of Type I Error (Sample size 100)



Assuming H_0 is True.

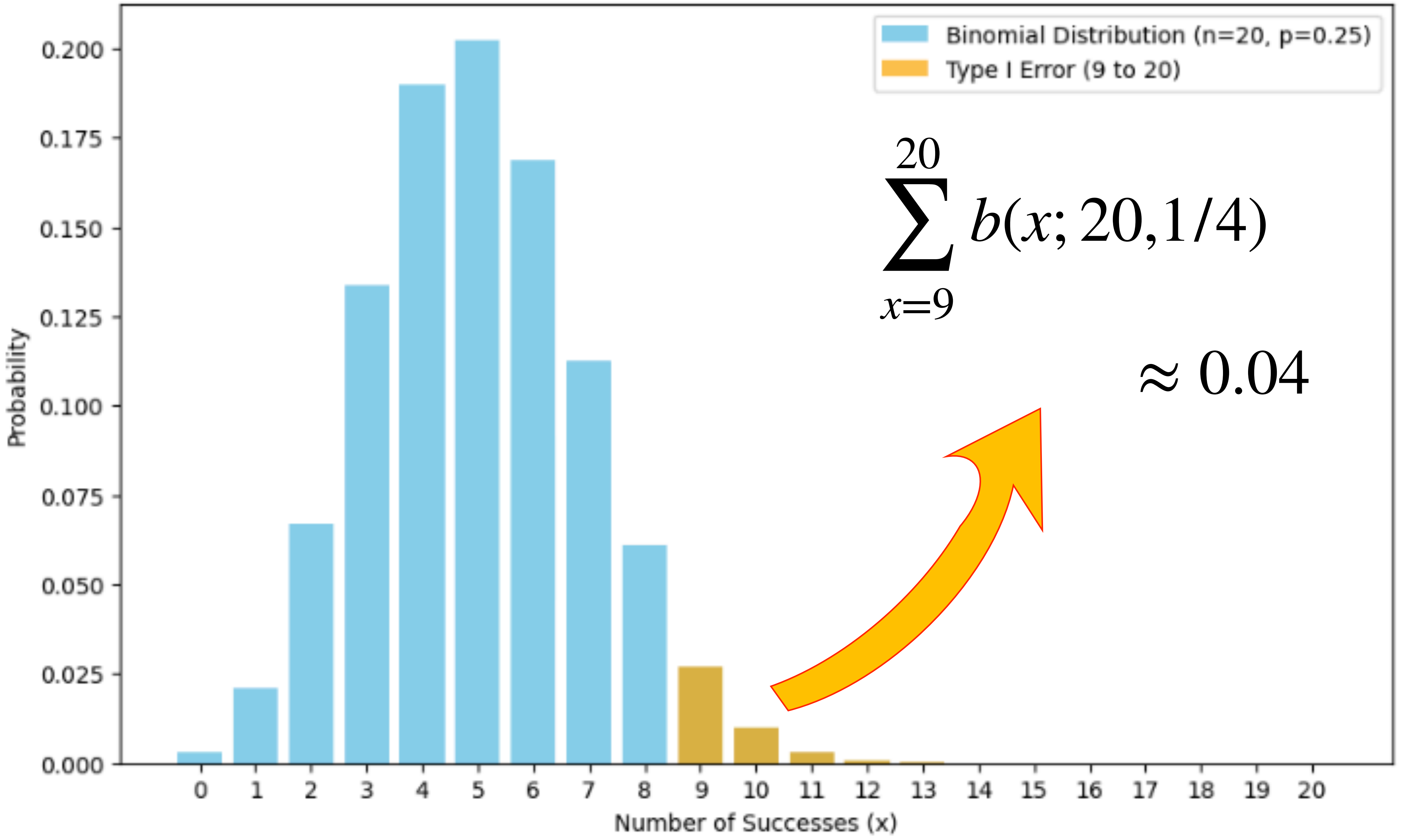
α Decreases as Sample Size Increases

Example: One-sided test with a discrete random variable

Testing whether a new vaccine has effectiveness more than $25\ \%$.

$$H_0 : \ p \le 0.25 \qquad H_1 : \ p > 0.25$$

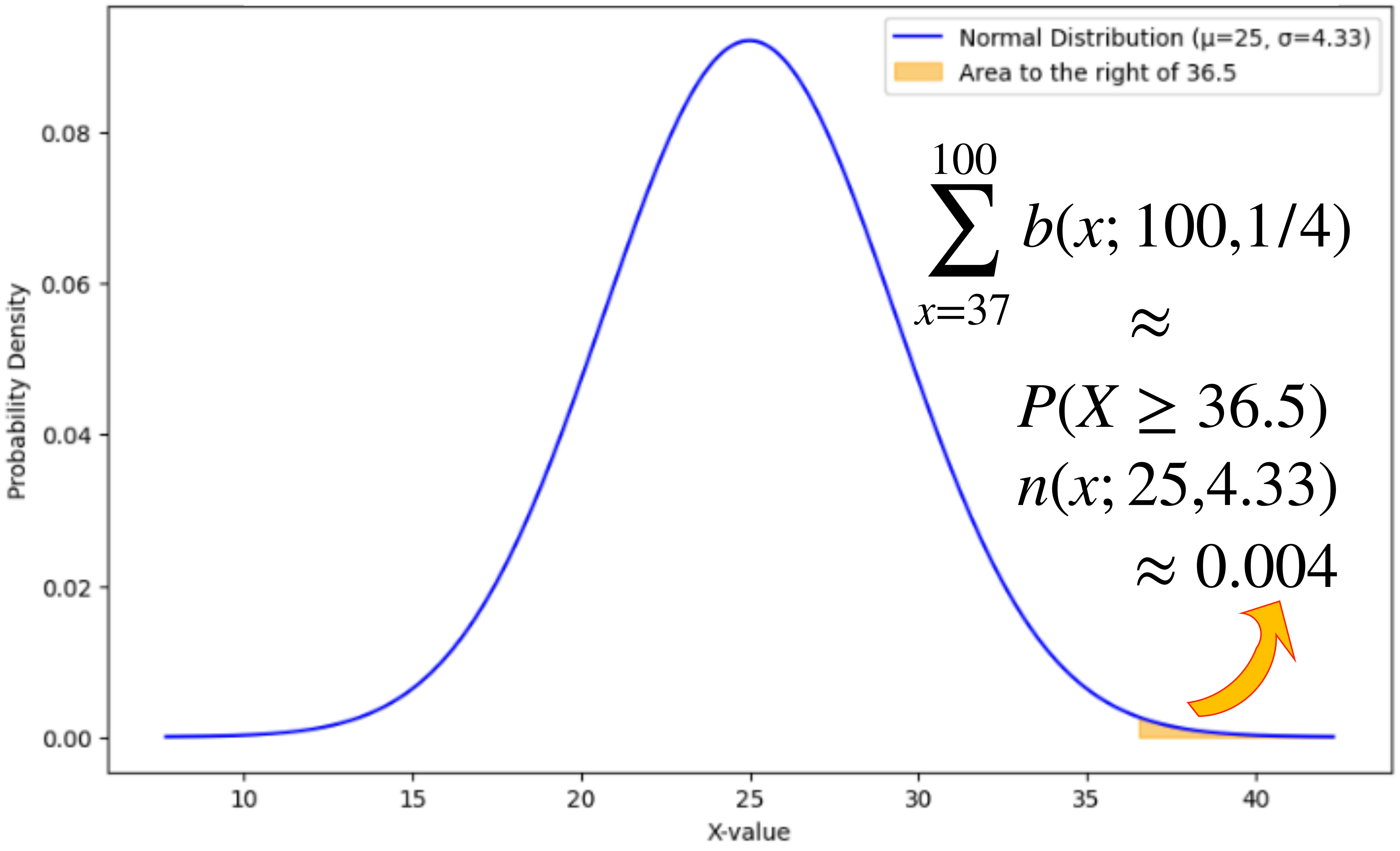
Probability of Type I Error (Sample size 20)



$$\sum_{x=9}^{20} b(x; 20, 1/4) \approx 0.04$$

Assuming H_0 is True.

Probability of Type I Error (Sample size 100)



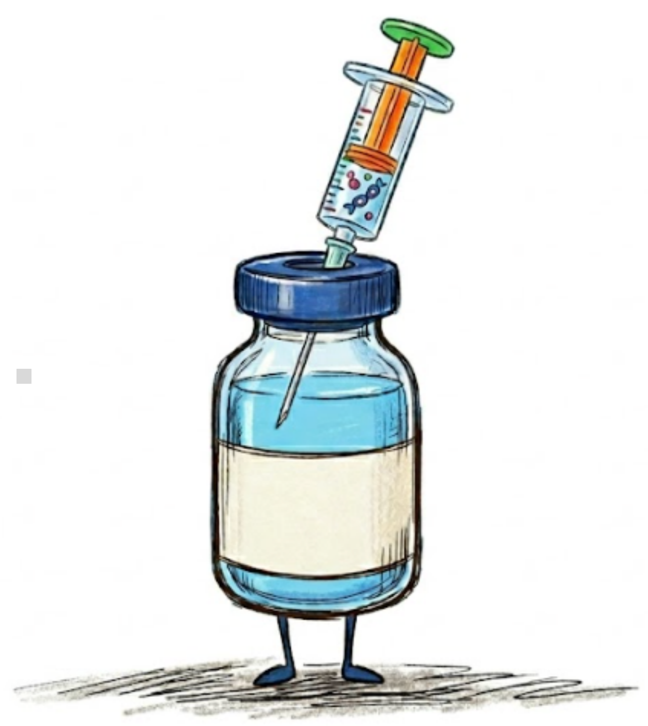
$$\sum_{x=37}^{100} b(x; 100, 1/4) \approx P(X \ge 36.5) \quad n(x; 25, 4.33) \approx 0.004$$

Assuming H_0 is True.

α Decreases as Sample Size Increases

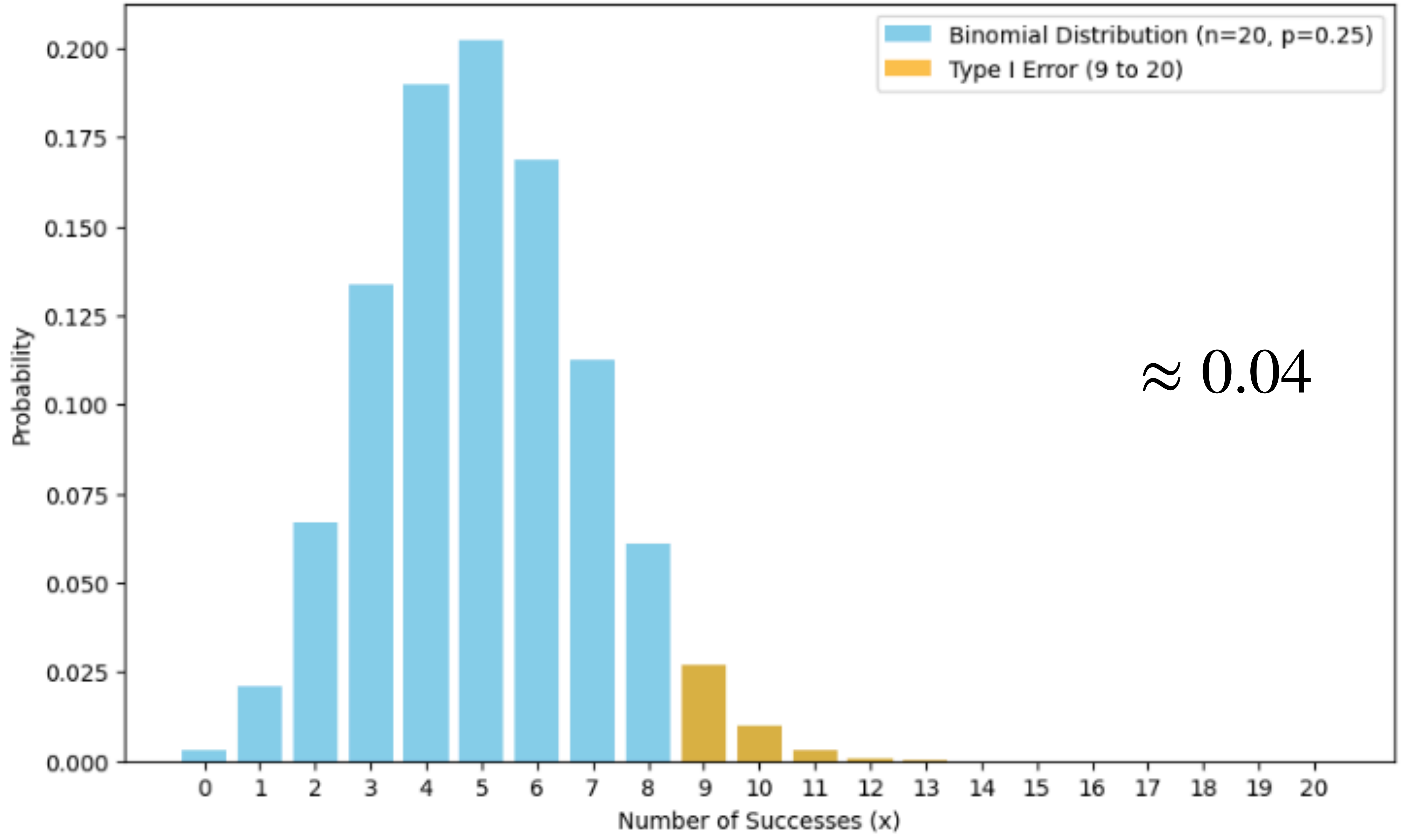
Example: One-sided test with a discrete random variable

Testing whether a new vaccine has effectiveness more than $25\ \%$.



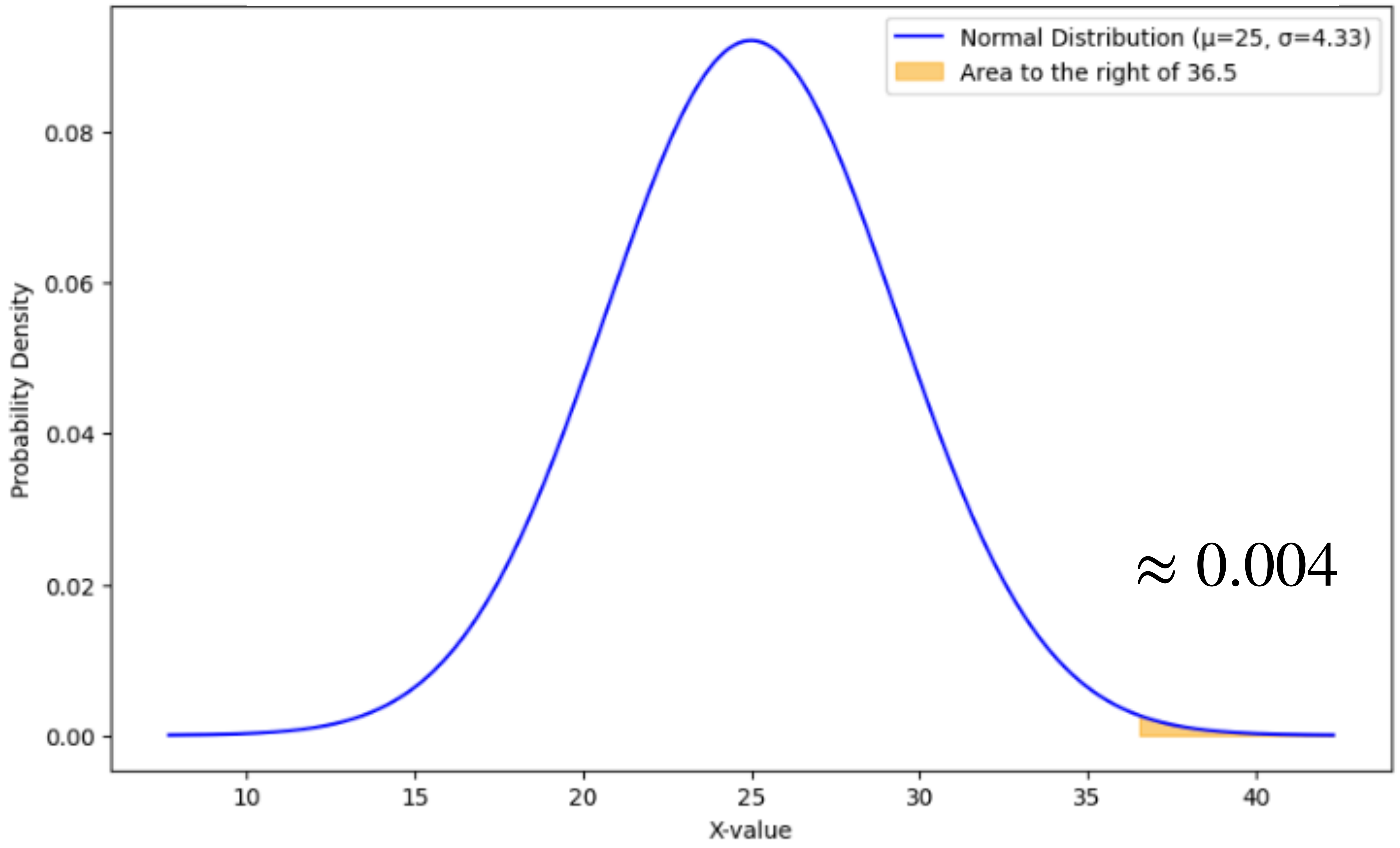
$$H_0 : \ p \leq 0.25 \qquad H_1 : \ p > 0.25$$

Probability of Type I Error (Sample size 20)



Assuming H_0 is True.

Probability of Type I Error (Sample size 100)



Assuming H_0 is True.

β Decreases as Sample Size Increases

Example: One-sided test with a discrete random variable

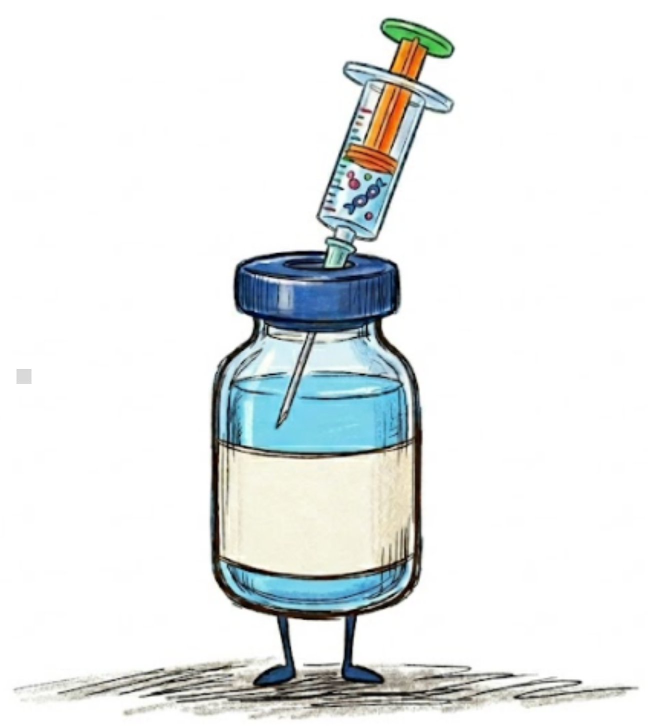
Testing whether a new vaccine has effectiveness more than $25\ \%$.

$$H_0 :\ p \leq 0.25 \qquad H_1 :\ p > 0.25$$

β Decreases as Sample Size Increases

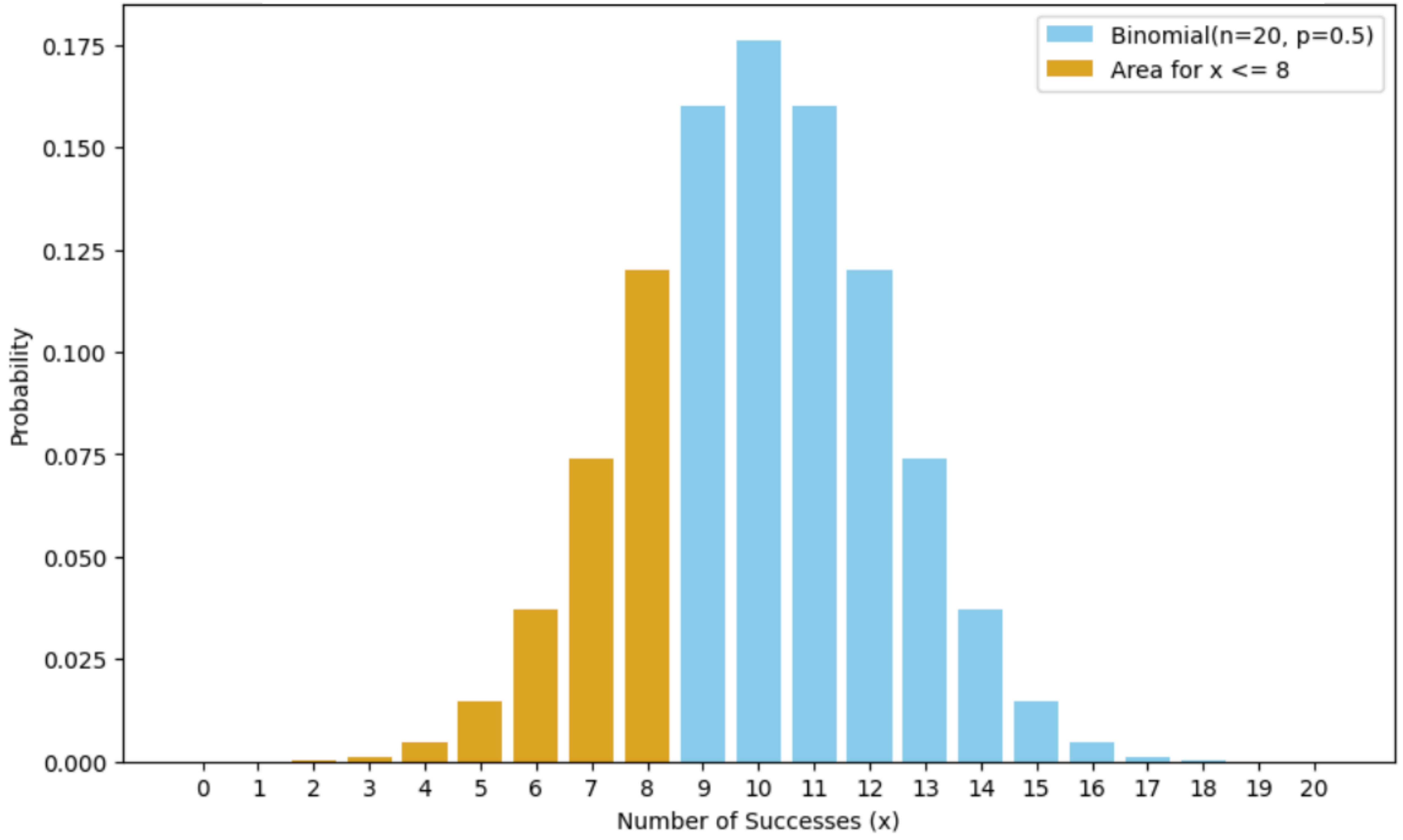
Example: One-sided test with a discrete random variable

Testing whether a new vaccine has effectiveness more than $25\ \%$.



$$H_0 :\ p \leq 0.25 \qquad H_1 :\ p > 0.25$$

Probability of Type II Error (Sample size 20)



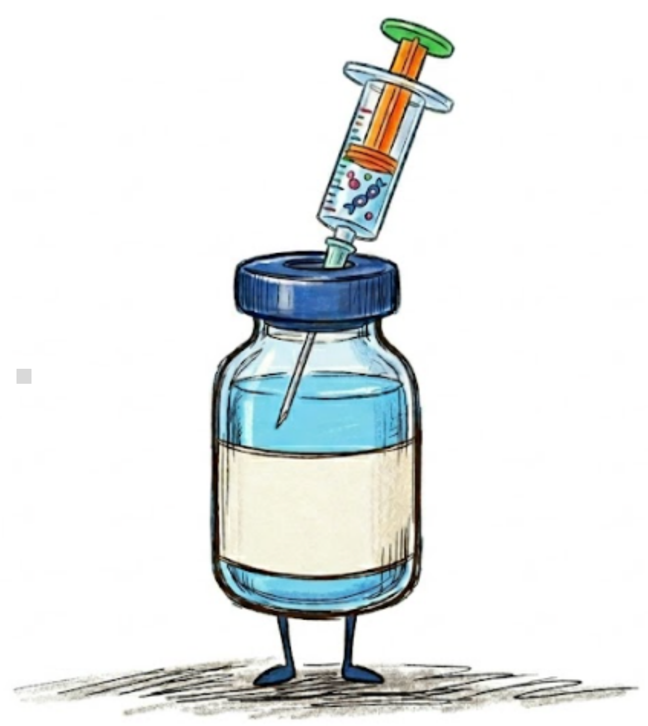
Assuming H_1 is True.

β Decreases as Sample Size Increases

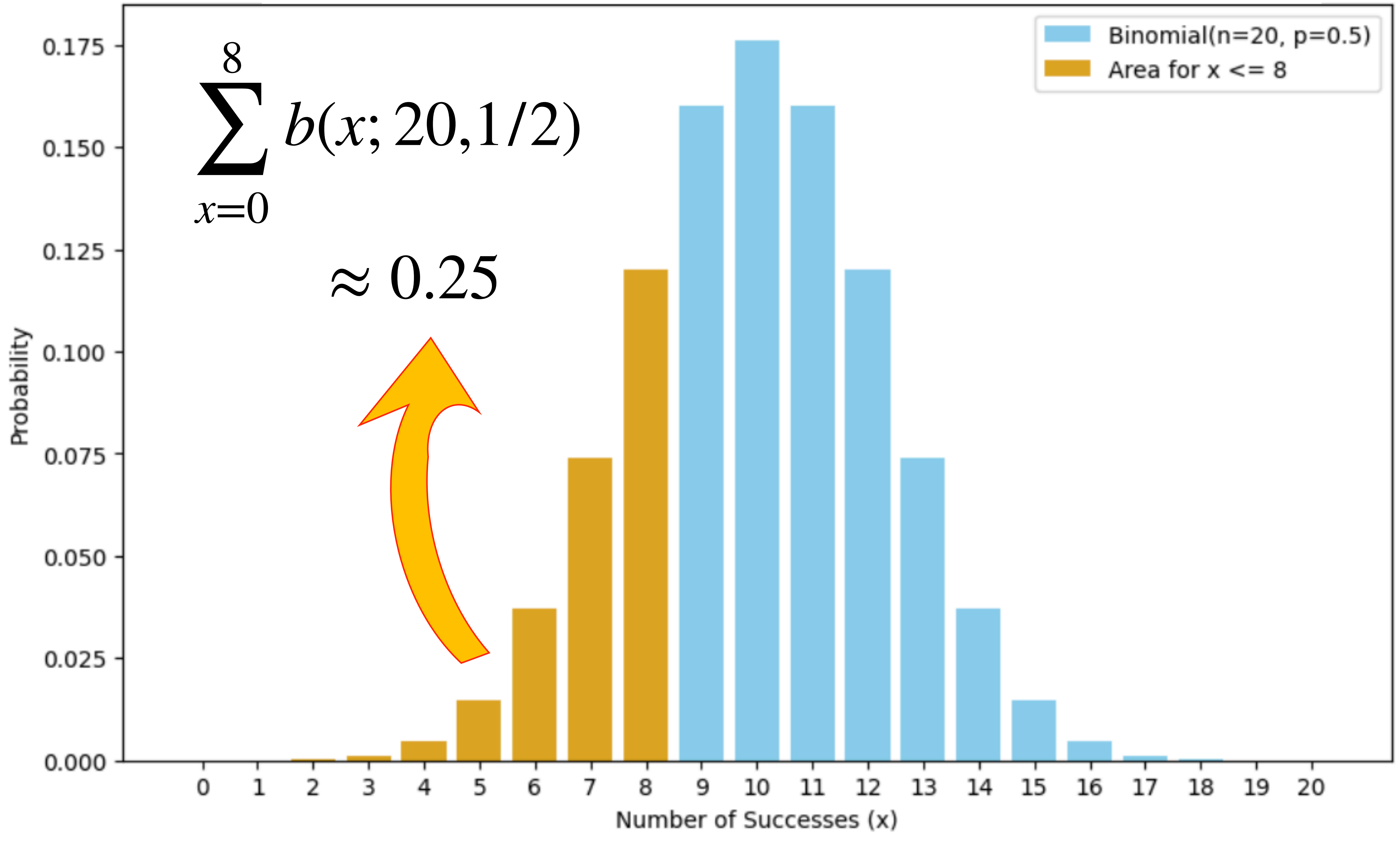
Example: One-sided test with a discrete random variable

Testing whether a new vaccine has effectiveness more than $25\ \%$.

$$H_0:\ p \leq 0.25 \qquad H_1:\ p > 0.25$$



Probability of Type II Error (Sample size 20)



Assuming H_1 is True.

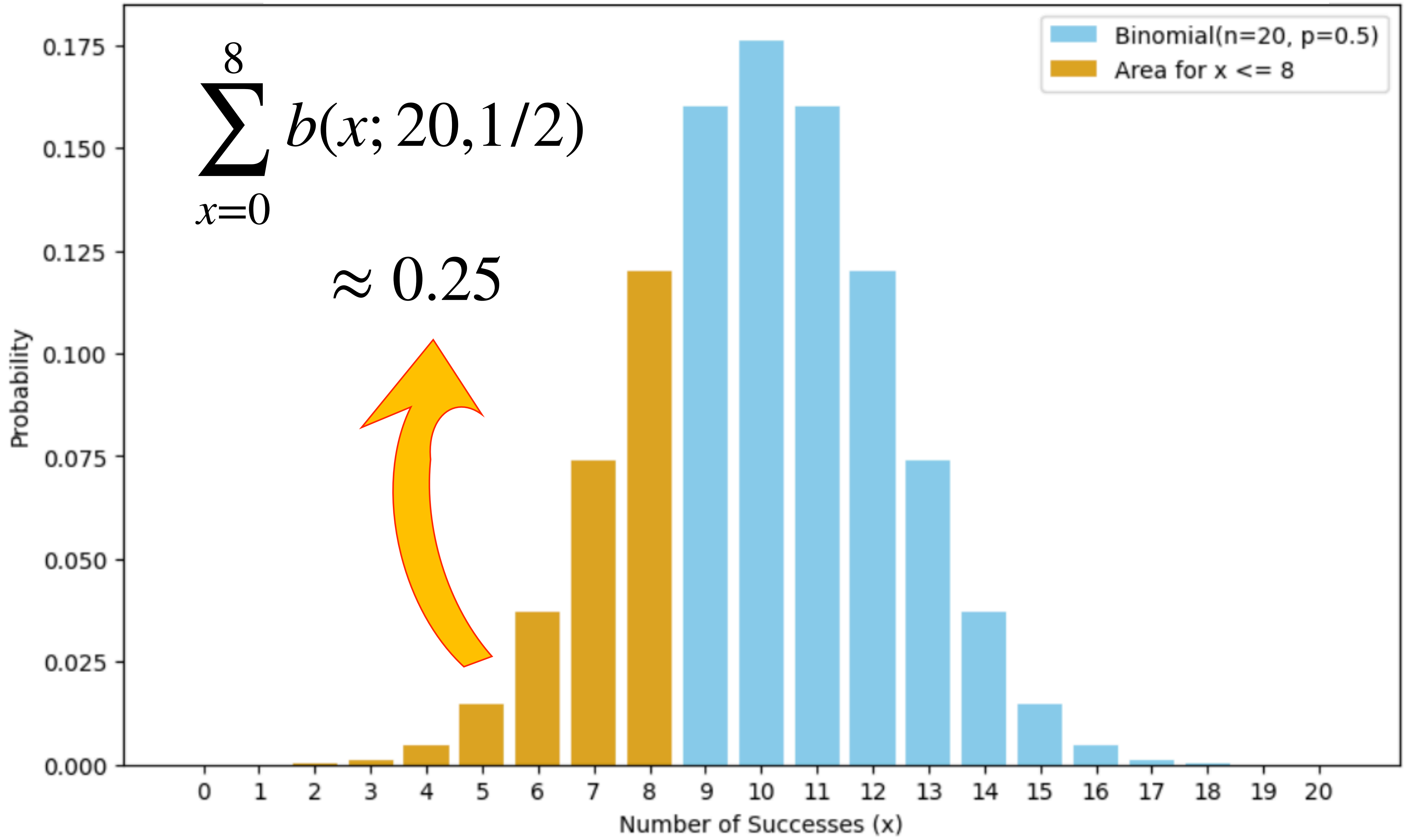
β Decreases as Sample Size Increases

Example: One-sided test with a discrete random variable

Testing whether a new vaccine has effectiveness more than 25 %.

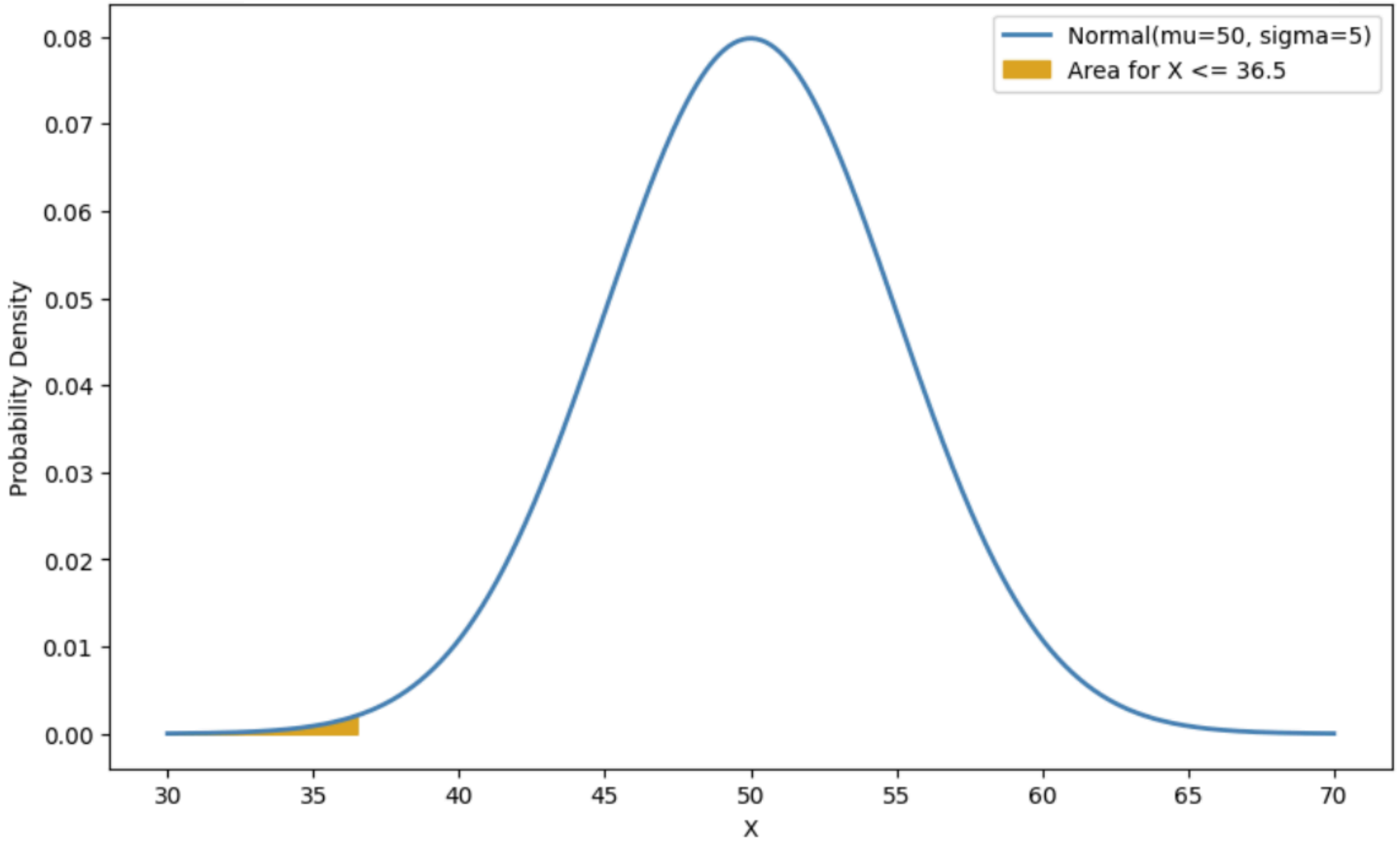
$$H_0: \ p \le 0.25 \qquad H_1: \ p > 0.25$$

Probability of Type II Error (Sample size 20)



Assuming H_1 is True.

Probability of Type II Error (Sample size 100)



Assuming H_1 is True.

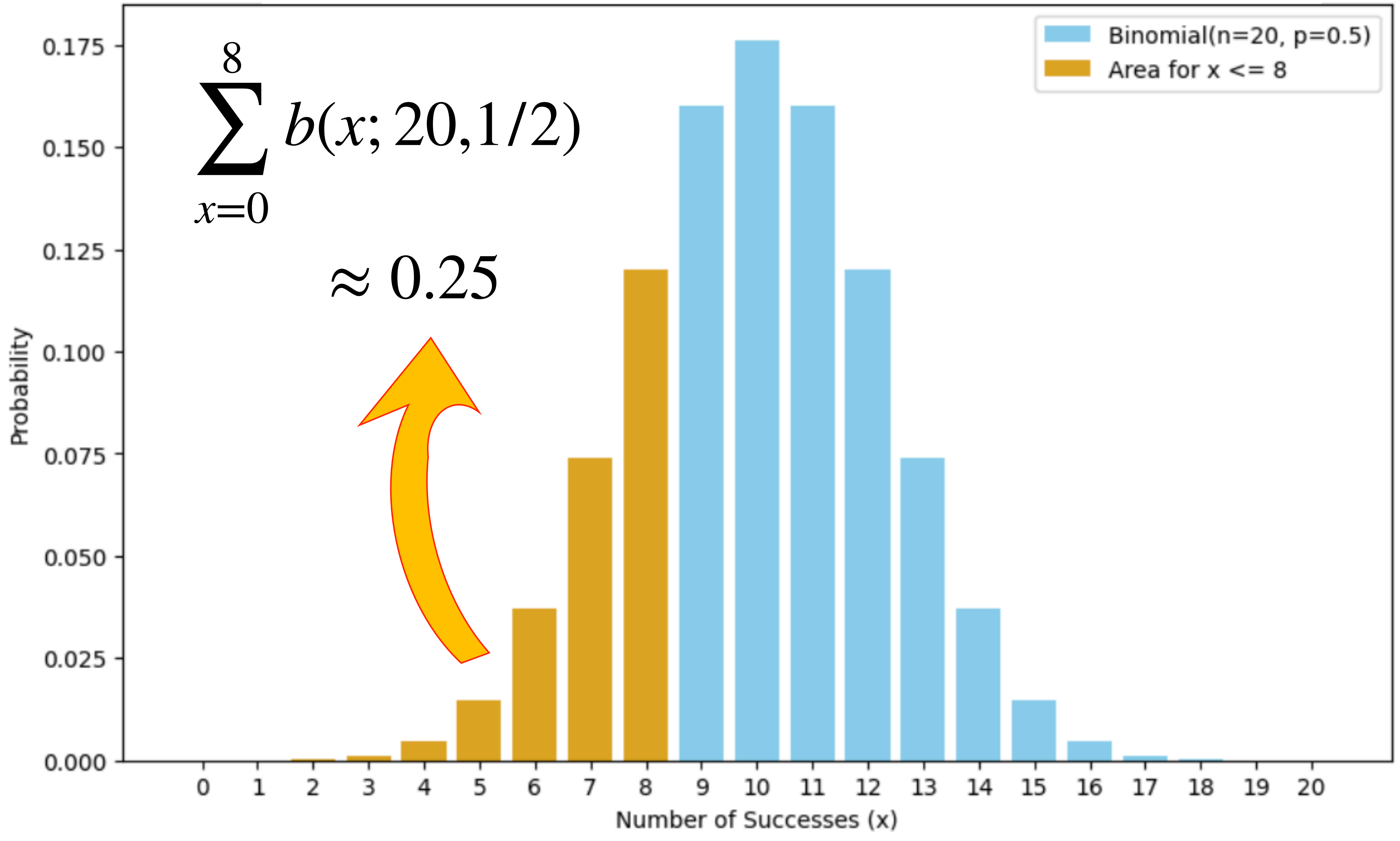
β Decreases as Sample Size Increases

Example: One-sided test with a discrete random variable

Testing whether a new vaccine has effectiveness more than $25\ \%$.

$$H_0:\ p \le 0.25 \qquad H_1:\ p > 0.25$$

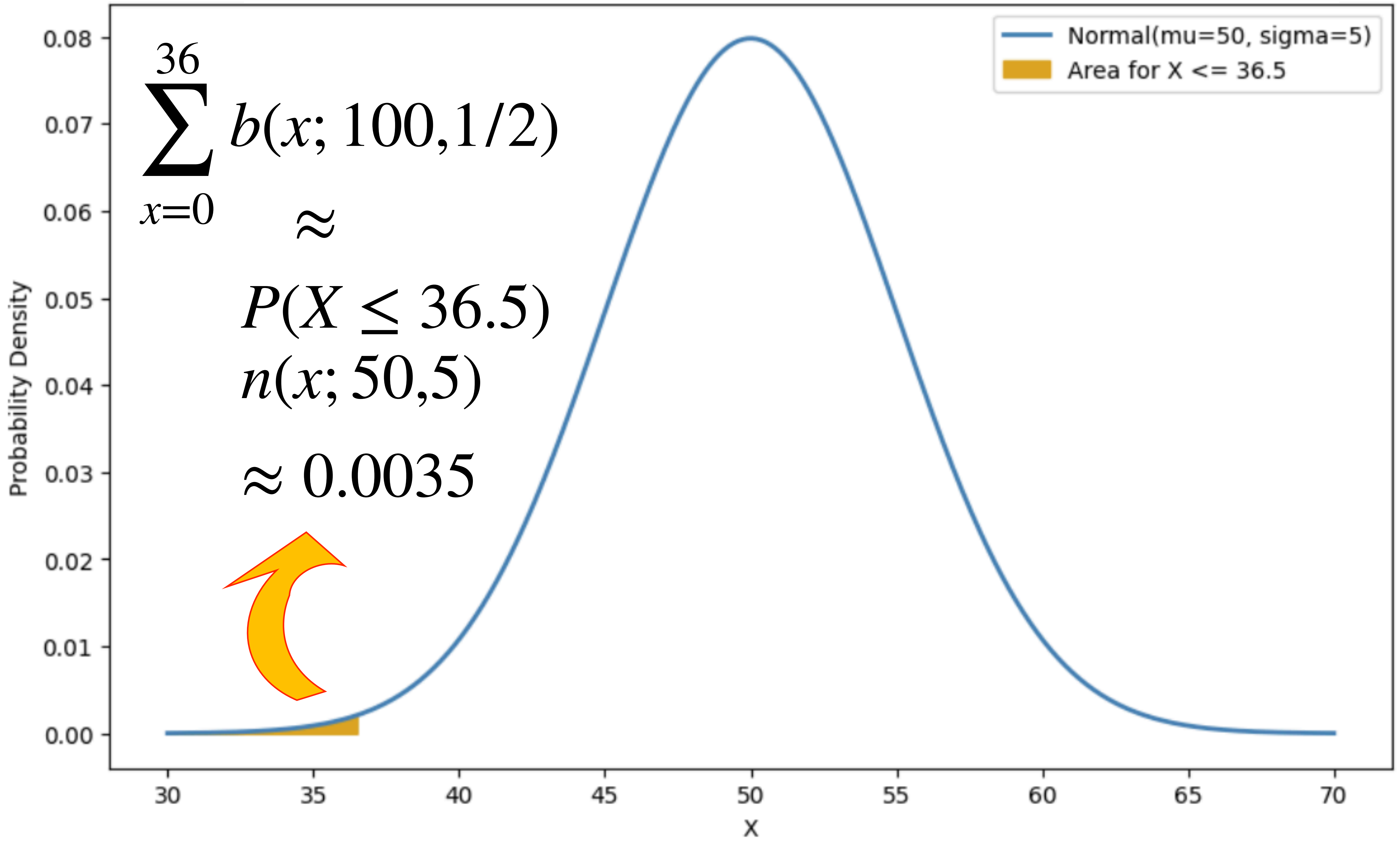
Probability of Type II Error (Sample size 20)



$$\sum_{x=0}^{8} b(x; 20, 1/2) \approx 0.25$$

Assuming H_1 is True.

Probability of Type II Error (Sample size 100)



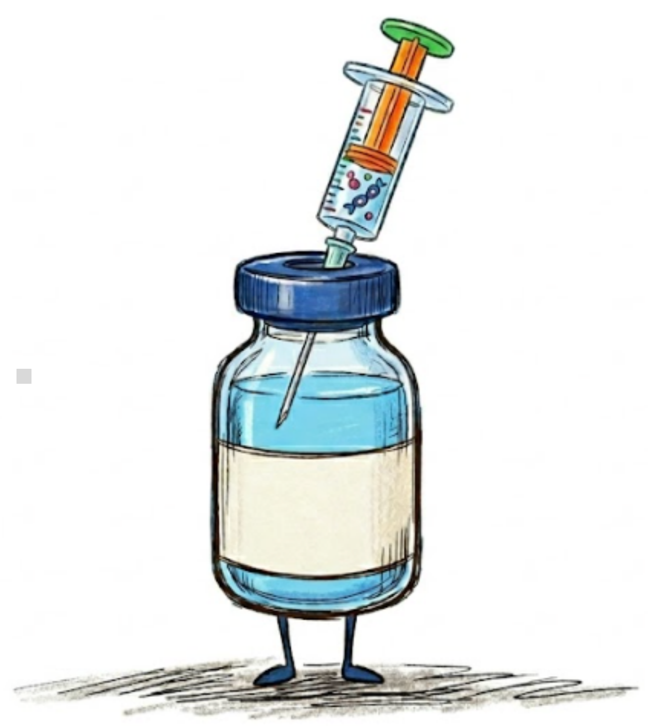
$$\sum_{x=0}^{36} b(x; 100, 1/2) \approx P(X \le 36.5)\ n(x; 50, 5) \approx 0.0035$$

Assuming H_1 is True.

β Decreases as Sample Size Increases

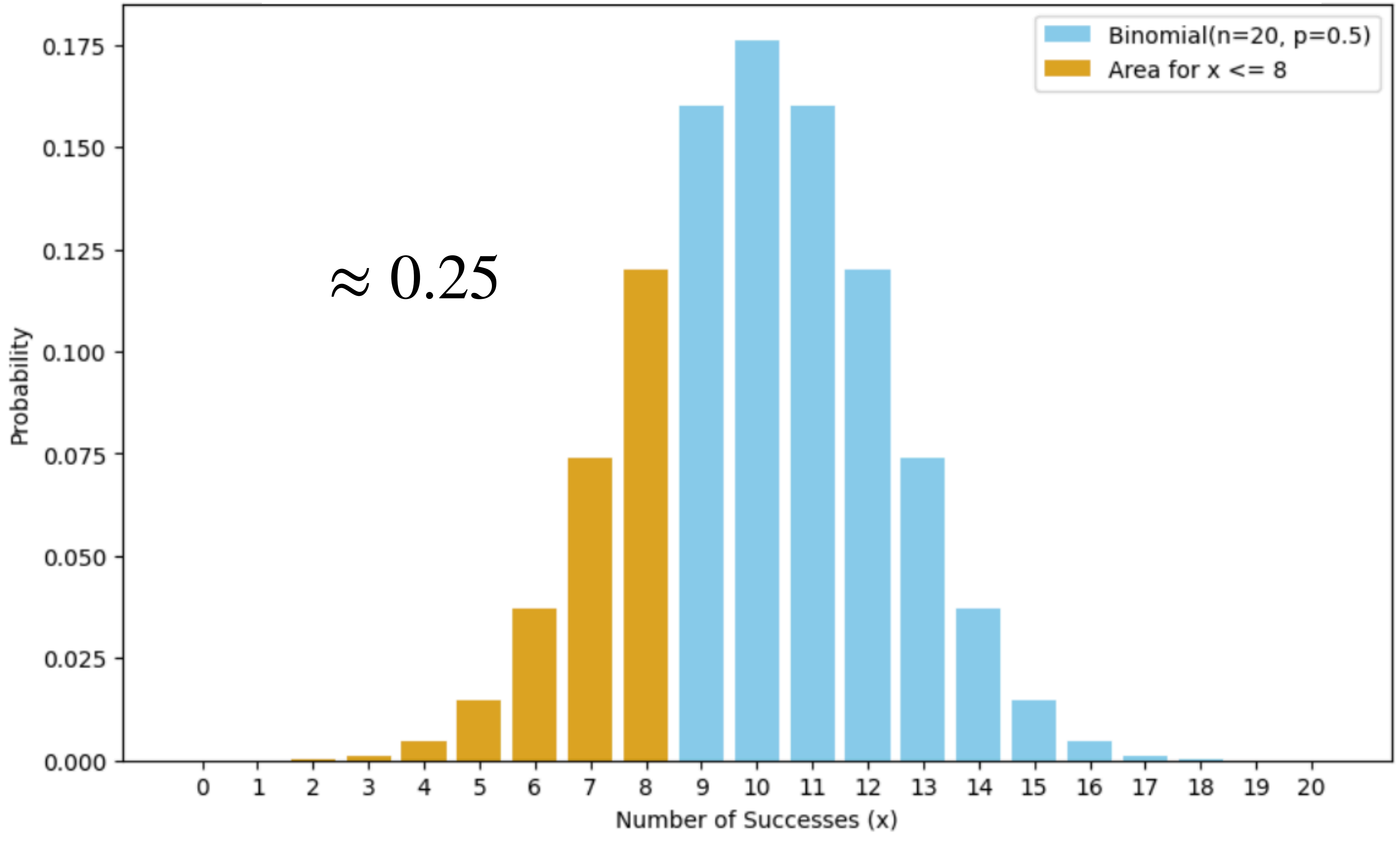
Example: One-sided test with a discrete random variable

Testing whether a new vaccine has effectiveness more than $25\ \%$.



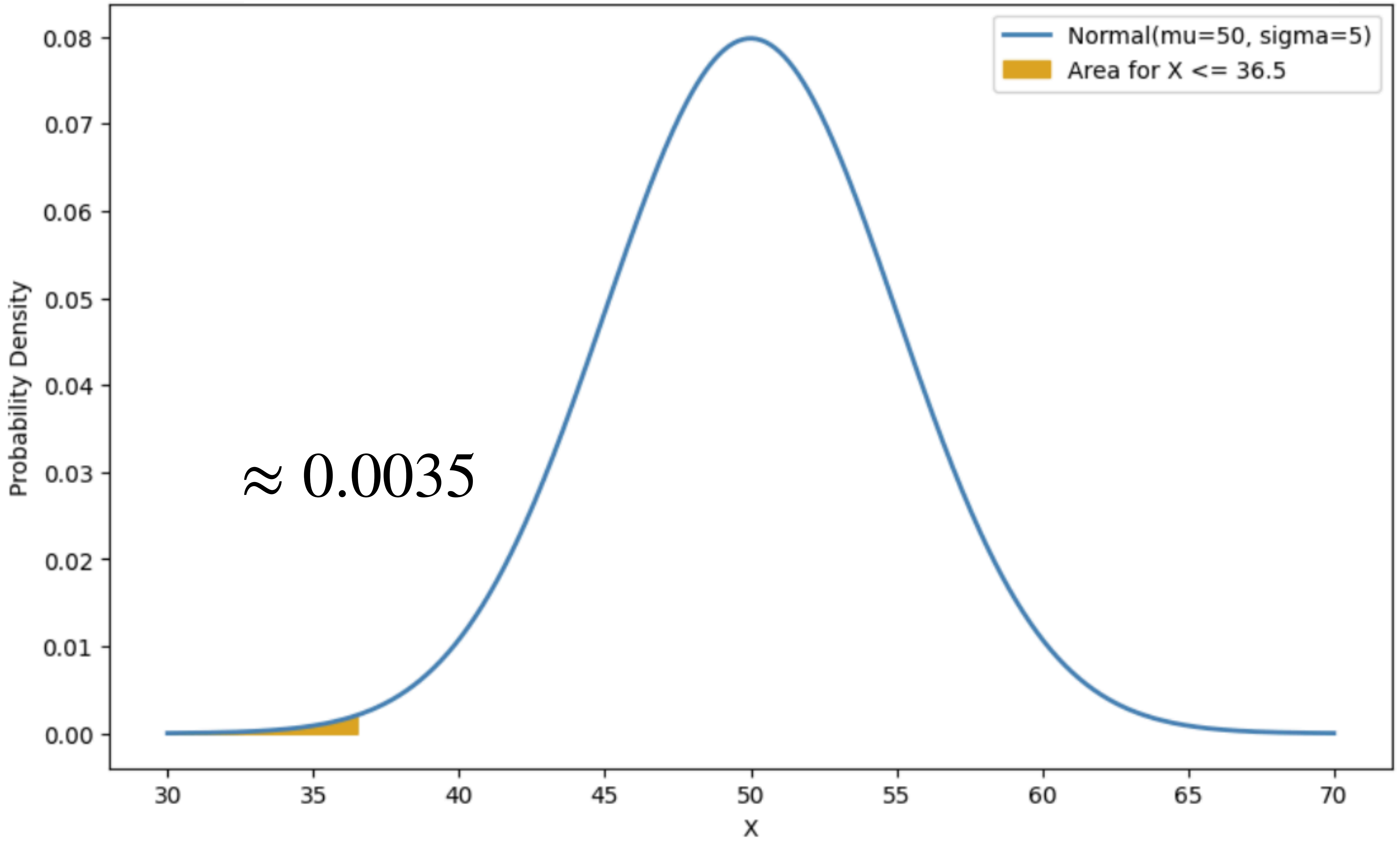
$$H_0 : \ p \leq 0.25 \qquad H_1 : \ p > 0.25$$

Probability of Type II Error (Sample size 20)



Assuming H_1 is True.

Probability of Type II Error (Sample size 100)



Assuming H_1 is True.

Illustration with a Continuous Random Variable

Example: Two-sided test with a continuous random variable

Testing whether average weight of male students is 68kg.

$$H_0 : \mu = 68 \qquad H_1 : \mu \neq 68$$

Assume $\sigma = 3.6$ and that we have a sample of size 36.

Illustration with a Continuous Random Variable

Example: Two-sided test with a continuous random variable

Testing whether average weight of male students is 68kg.

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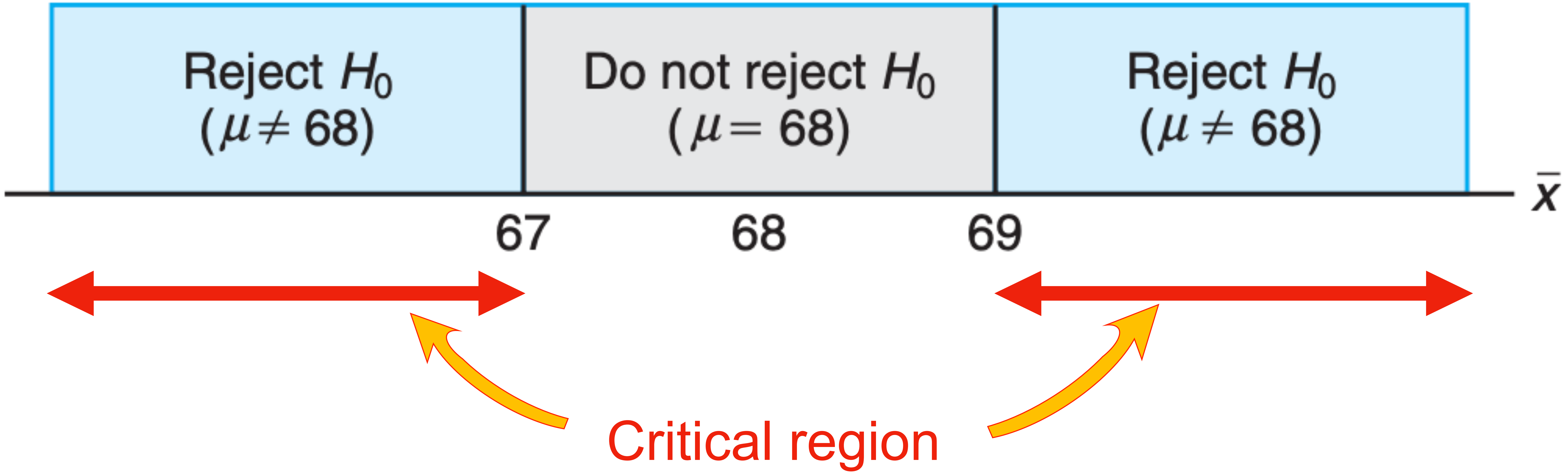
Assume $\sigma = 3.6$ and that we have a sample of size 36.

We had decided to consider the following critical region:



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Testing whether average weight of male students is 68kg.

$$H_0 : \mu = 68 \qquad H_1 : \mu \neq 68$$

Assume $\sigma = 3.6$ and that we have a sample of size 36.

Assuming H_0 is True.

What is $\alpha = P$(type I error)?

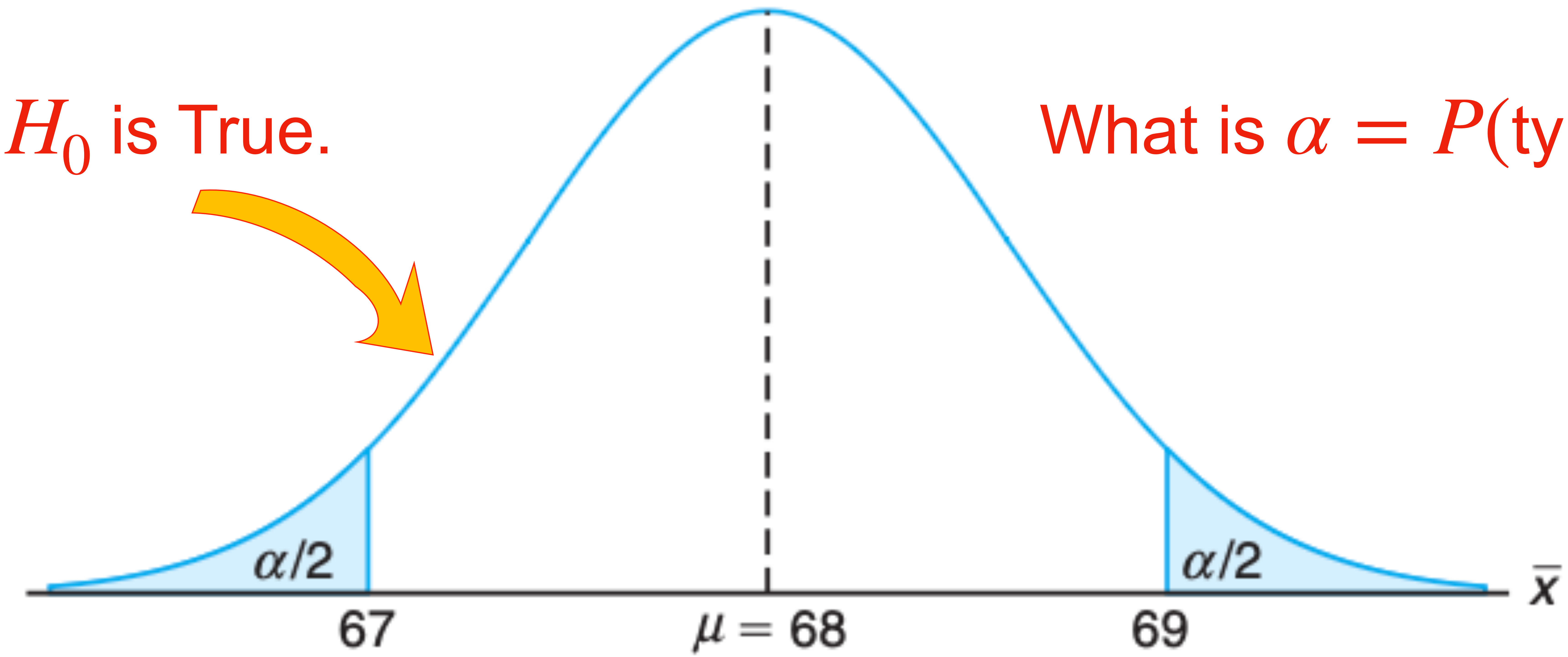

Illustration with a Continuous Random Variable

Example: Two-sided test with a continuous random variable

Testing whether average weight of male students is 68kg.

$$H_0: \ \mu = 68 \qquad H_1: \ \mu \neq 68$$

Assume $\sigma = 3.6$ and that we have a sample of size 36.

Use $norm.cdf(-1.67) = 0.0475$

$$\alpha = P(\text{type I error}) = P(\bar{X} < 67 \text{ when } \mu = 68) + P(\bar{X} > 69 \text{ when } \mu = 68)$$

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Testing whether average weight of male students is 68kg.

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Assume $\sigma = 3.6$ and that we have a sample of size 36.

Use $norm.cdf(-1.67) = 0.0475$

$$\alpha = P(\text{type I error}) = P(\bar{X} < 67 \text{ when } \mu = 68) + P(\bar{X} > 69 \text{ when } \mu = 68)$$

$$z_1 = \frac{67 - 68}{0.6} = -1.67 \text{ and } z_2 = \frac{69 - 68}{0.6} = 1.67$$

$$\alpha = P(Z < -1.67) + P(Z > 1.67) = 2P(Z < -1.67) = 0.095$$

Illustration with a Continuous Random Variable

Example: Two-sided test with a continuous random variable

Testing whether average weight of male students is 68kg.

$$H_0 : \ \mu = 68 \qquad H_1 : \ \mu \neq 68$$

Assume $\sigma = 3.6$ and that we have a sample of size 36.

To reduce α, we can increase the sample size, say to 64.

$$\alpha = P(Z < -1.67) + P(Z > 1.67) = 2P(Z < -1.67) = 0.095$$

Illustration with a Continuous Random Variable

Example: Two-sided test with a continuous random variable

Testing whether average weight of male students is 68kg.

$$H_0 : \ \mu = 68 \qquad H_1 : \ \mu \neq 68$$

Assume $\sigma = 3.6$ and that we have a sample of size 64.

Use $norm.cdf(-2.22) = 0.013$

$$\alpha = P(\text{type I error}) = P(\bar{X} < 67 \text{ when } \mu = 68) + P(\bar{X} > 69 \text{ when } \mu = 68)$$

$$z_1 = \frac{67 - 68}{0.45} = -2.22 \text{ and } z_2 = \frac{69 - 68}{0.45} = 2.22$$

$$\alpha = P(Z < -2.22) + P(Z > 2.22) = 2P(Z < -2.22) = 0.026$$

Illustration with a Continuous Random Variable

Example: Two-sided test with a continuous random variable

Testing whether average weight of male students is 68kg.

$$H_0: \ \mu = 68 \qquad\qquad H_1: \ \mu \neq 68$$

Assume $\sigma = 3.6$ and that we have a sample of size 64.

Let's see β and Type II error. We need specific alternative hypothesis μ.

H_0 is not rejected even though $\mu = 70$.

Illustration with a Continuous Random Variable

Example: Two-sided test with a continuous random variable

Testing whether average weight of male students is 68kg.

$$H_0: \ \mu = 68 \qquad H_1: \ \mu \neq 68$$

Assume $\sigma = 3.6$ and that we have a sample of size 64.

$\beta = P(\text{type II error})$

Assuming H_1 is True.

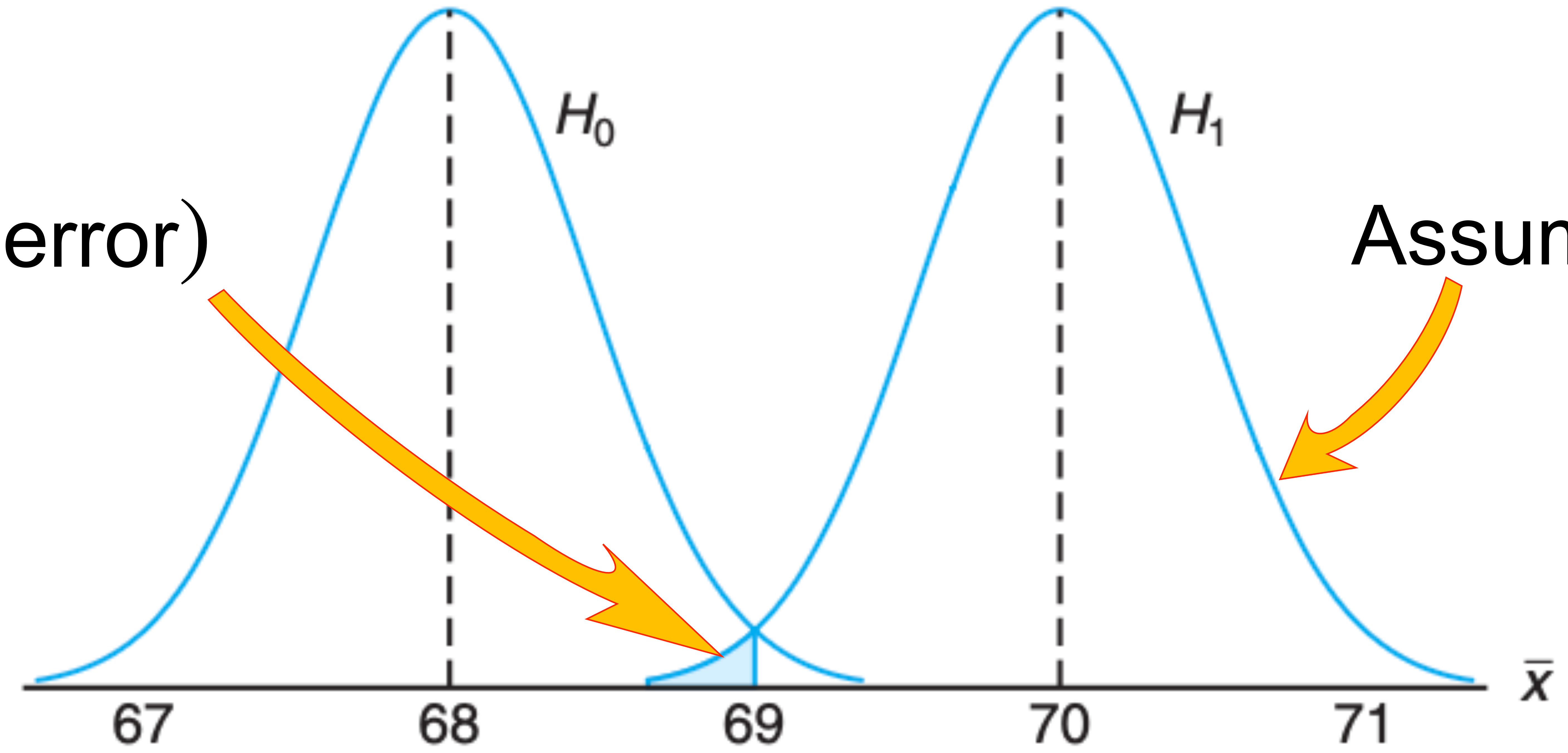

Illustration with a Continuous Random Variable

Example: Two-sided test with a continuous random variable

Testing whether average weight of male students is 68kg.

$$H_0: \ \mu = 68 \qquad H_1: \ \mu \neq 68$$

Assume $\sigma = 3.6$ and that we have a sample of size 64.

Use $norm.cdf(-2.22) \approx 0.013$

Use $norm.cdf(-6.67) \approx 0.0$

$\beta = P(\text{type II error}) = P(67 \leq \bar{X} \leq 69 \text{ when } \mu = 70)$

$z_1 = \dfrac{67 - 70}{0.45} = -6.67$ and $z_2 = \dfrac{69 - 70}{0.45} = -2.22$

$$\beta = P(-6.67 \leq Z \leq -2.22) = P(Z \leq -2.22) - P(Z \leq -6.67) \approx 0.013$$

Illustration with a Continuous Random Variable

The **power** of a test is the probability of rejecting H_0 given that a specific alternative is true.

Power of test $= 1 - \beta$

Illustration with a Continuous Random Variable

Example: Two-sided test with a continuous random variable

Testing whether average weight of male students is 68kg.

$$H_0 : \mu = 68 \qquad H_1 : \mu \neq 68$$

Assume $\sigma = 3.6$ and that we have a sample of size 64.

$\beta = P(\text{type II error}) = P(67 \leq \bar{X} \leq 69 \text{ when } \mu = 70)$

Power of test is high.

$z_1 = \dfrac{67 - 70}{0.45} = -6.67$ and $z_2 = \dfrac{69 - 70}{0.45} = -2.22$

$$\beta = P(-6.67 \leq Z \leq -2.22) = P(Z \leq -2.22) - P(Z \leq -6.67) = 0.013$$

Illustration with a Continuous Random Variable

Example: Two-sided test with a continuous random variable

Testing whether average weight of male students is 68kg.

$$H_0: \ \mu = 68 \qquad H_1: \ \mu \neq 68$$

Assume $\sigma = 3.6$ and that we have a sample of size 64.

Power of test decreases with specific alternative close to H_0, say $\mu = 68.5$:

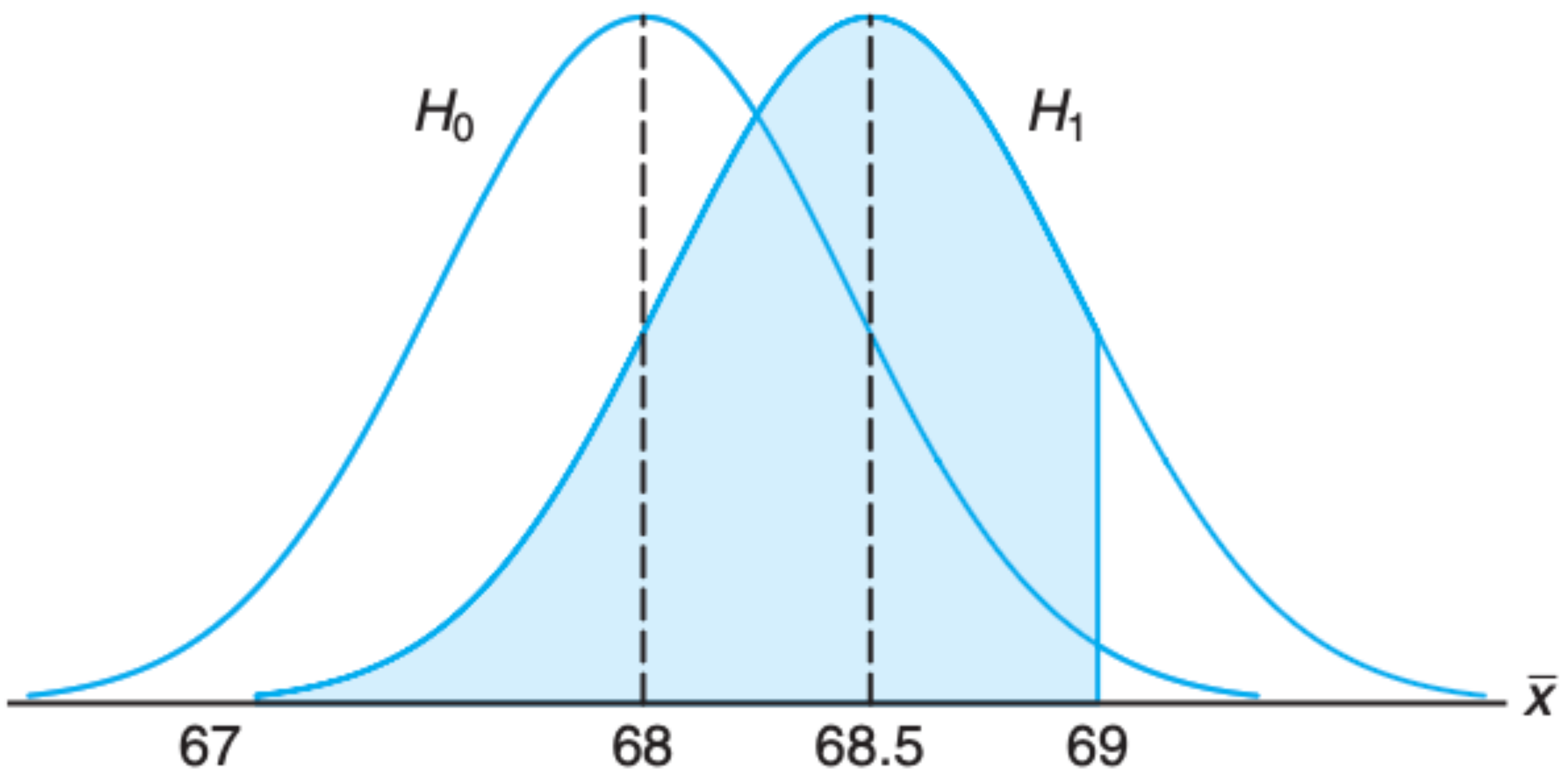

Illustration with a Continuous Random Variable

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Power of test decreases with specific alternative close to H_0, say $\mu = 68.5$:

$\beta = P(\text{type II error})$

Assuming H_1 is True.

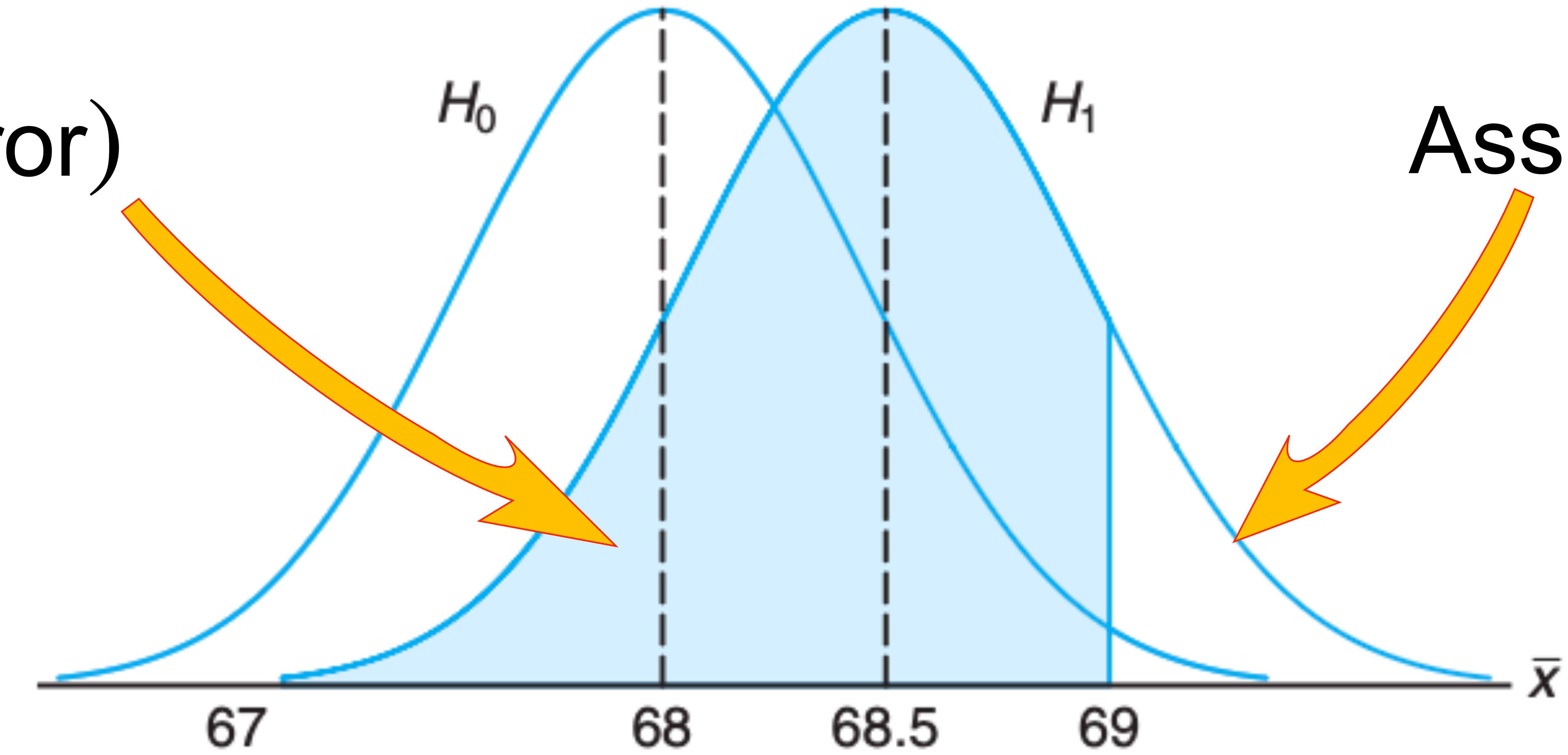

Illustration with a Continuous Random Variable

Example: Two-sided test with a continuous random variable

Testing whether average weight of male students is 68kg.

$$H_0: \ \mu = 68 \qquad H_1: \ \mu \neq 68$$

Assume $\sigma = 3.6$ and that we have a sample of size 64.

$\beta = P(\text{type II error}) = P(67 \leq \bar{X} \leq 69 \text{ when } \mu = 68.5)$

Power of test is low.

$$z_1 = \frac{67 - 68.5}{0.45} = -3.33 \text{ and } z_2 = \frac{69 - 68.5}{0.45} = 1.11$$

$$\beta = P(-3.33 \leq Z \leq 1.11) = P(Z \leq 1.11) - P(Z \leq -3.33) \approx 0.87$$