CSC196: Analyzing Data

Sampling Distributions

Jason Pacheco and Cesim Erten

QUIZ

Draw two normal distributions with the same mean $\mu_1 = \mu_2$ and $\sigma_1 < \sigma_2$ on the same plot.

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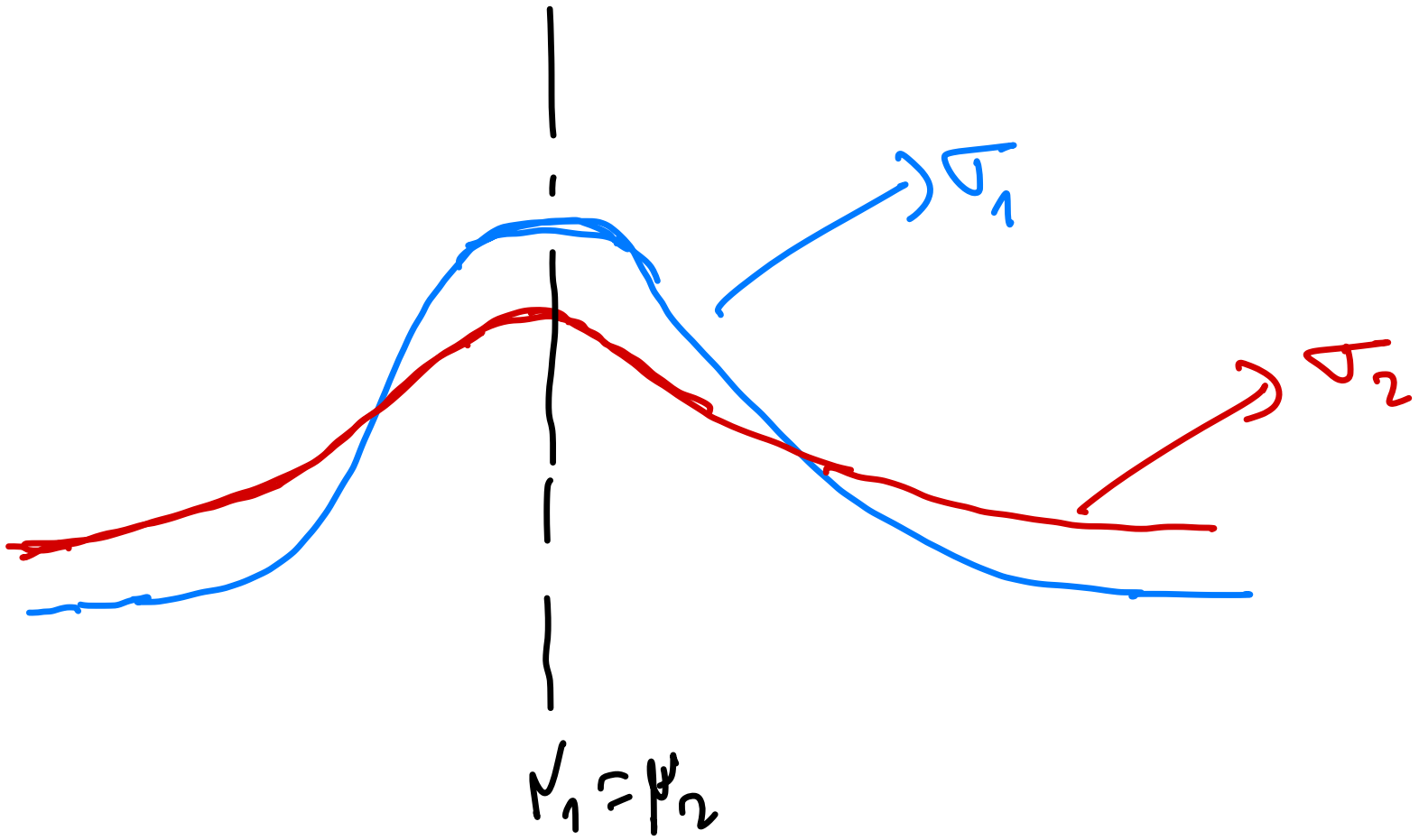
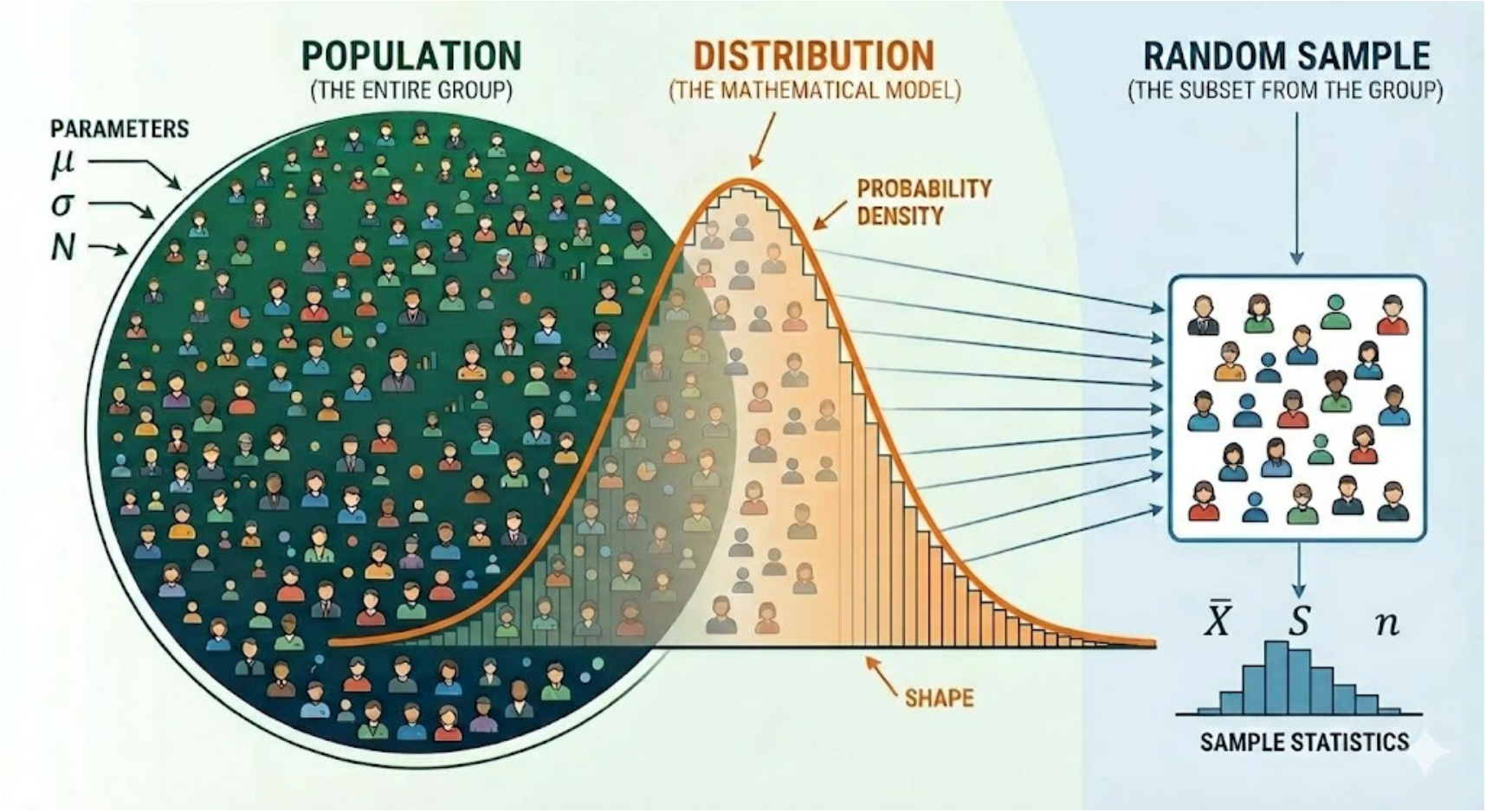


Outline

- Random Sampling
- Sampling Distributions
- Central Limit Theorem and Sampling Distribution of Means

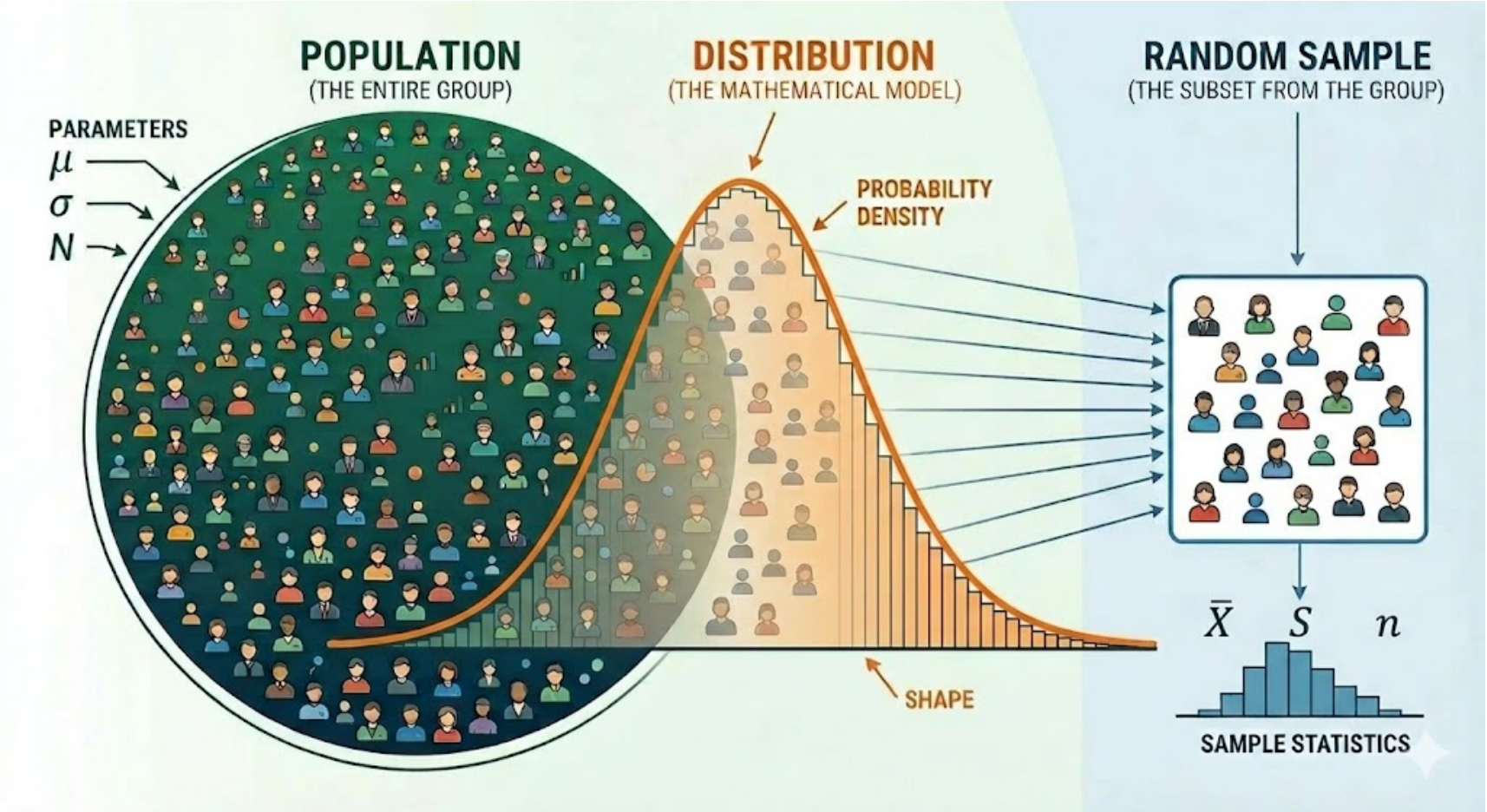
Random Sampling



A population is the totality of the observations with which we are concerned.

"Normal population" $\equiv$ A population whose observations are values of a random variable having a normal distribution

Random Sampling



A sample is a subset of a population.

In a random sample observations are made independently and at random:

Independent random variables $X_1, X_2, \cdots, X_n$ each with the same probability distribution $f(x)$.

Random Sampling

Any function of the random variables constituting a random sample is called a statistic. Let's review some statistics from initial lectures:

Let $X_1, \cdots, X_n$ iid (independently, identically distributed) random variables:

Sample mean: $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$

Sample median: $\tilde{x} = \begin{cases} x_{(n+1)/2}, & \text{if n is odd} \\ \frac{1}{2}(x_{n/2} + x_{n/2+1}), & \text{otherwise} \end{cases}$

Sample mode: Value in the sample that occurs the most.

Random Sampling

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Let $X_1, \cdots, X_n$ iid (independently, identically distributed) random variables:

Sample variance: $S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \overline{X})^2$

Sample standard deviation: $S = \sqrt{S^2}$

Sample range: $X_{max} - X_{min}$

Sampling Distributions and Sampling Distribution of Means

Probability distribution of a statistic is called a sampling distribution.

Let $X_1, \cdots, X_n$ iid with mean μ and variance σ^2.

Probability distribution of $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ is sampling distribution of the mean.

Note that $\overline{X}$ is a random variable:

$E(\overline{X}) =$

$Var(\overline{X}) =$

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$Var(\overline{X}) =$

(same as distribution mean μ)

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Remember:
$Var(aX) = a^2 Var(X)$

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Remember: X_i are independent

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In summary =

Expected value of sample mean is the same as the mean of the underlying distribution (and is not related to the size of sample, n)

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Variance of sample mean is inversely proportional to the size of sample, n.

Examples Combining Sample Mean with Chebyshev's

Review: X a random variable $\Rightarrow \forall\, t > 0,\ P(\,|X - \mu| \geq t) \leq \dfrac{\sigma^2}{t^2}$

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$$P(\mu - k\sigma < X < \mu + k\sigma) \geq 1 - \frac{1}{k^2}.$$

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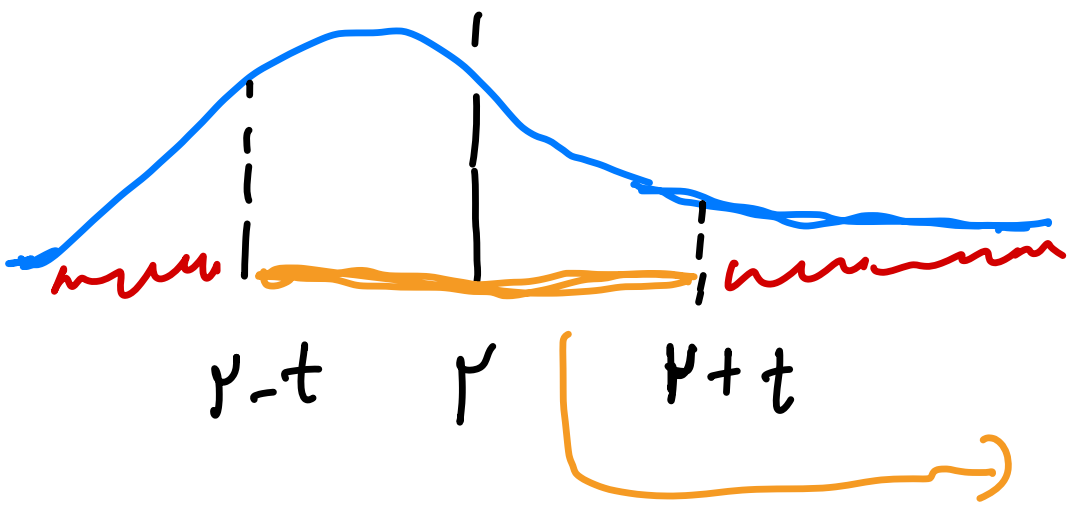

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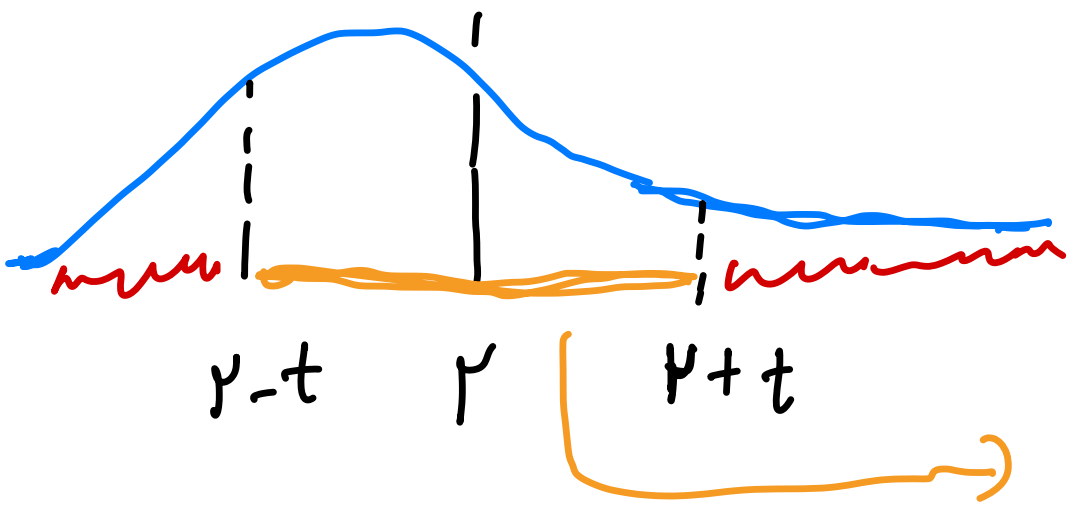

Replacing t with kσ gives us the exact special form.

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Example: Poll $100{,}000$ people. Assume each person in the population votes for candidate C independently with probability p.

Bound probability that poll is off by $\geq 1\,\%$?

Use Chebyshev's Theorem.

First find maximum possible variance of a Bernoulli RV.

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$Var(X_i) = p(1-p) \leq 0.25$

Rectangle with sides p and $(1-p)$ has maximum area when $p = (1-p)$

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OR equivalently: find p where derivative of $p - p^2$ is 0.

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$\Rightarrow Var(\bar{X}) \leq \frac{0.25}{100{,}000}$

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$\Rightarrow Var(\bar{X}) \leq \frac{0.25}{100.000}$

Applying Chebyshev's =

$P((\bar{X} - p) \geq 0.01) \leq \frac{0.25}{100.000} \times \frac{1}{(0.01)^2} = 2.5\%$