Computer Science

CSC196: Analyzing Data

Sampling Distributions

Jason Pacheco and Cesim Erten

Review of Previous Lecture

Let $X_1, \cdots, X_n$ iid with mean μ and variance σ^2.

Probability distribution of $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ is sampling distribution of the mean.

$E(\overline{X}) =$ $\qquad$ $Var(\overline{X}) =$

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$E(\overline{X}) = \mu$ $\qquad$ $Var(\overline{X}) = \dfrac{\sigma^2}{n}$

Outline

- Sampling Distributions (continued)
- Central Limit Theorem

Examples Combining Sample Mean with Chebyshev's

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Size of sample such that $\overline{X}$ is within 2 units of μ with ≥ 0.99 probability?

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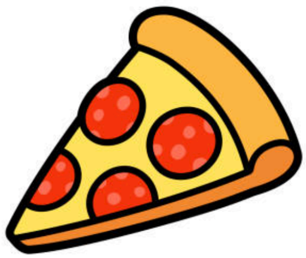
$$\Rightarrow 1 - \frac{2}{n} = 0.99 \Rightarrow n = \frac{2}{0.01} = \underline{\underline{200}}$$

Central Limit Theorem

Example: Time it takes to deliver a pizza: exponential distribution



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Consider the average delivery times of 30 drivers over a week:

- Individual delivery times might be erratic.
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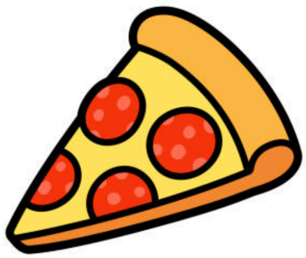

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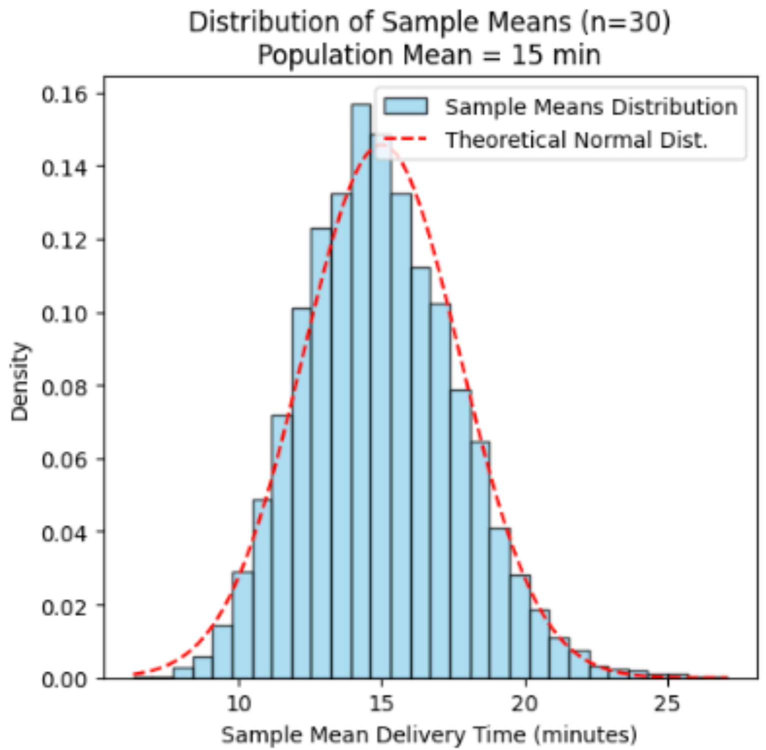

Check: https://colab.research.google.com/drive/1DOHaH6GgxSQkgLQLkR1xpNBcw98lEcp4?usp=sharing

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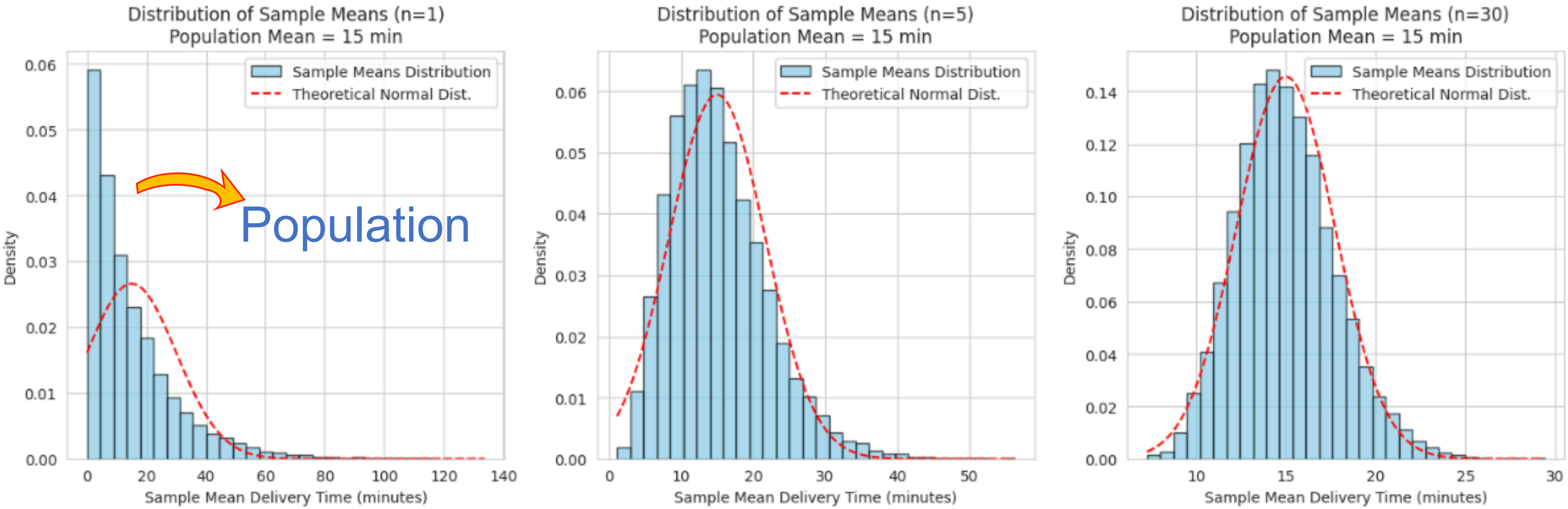

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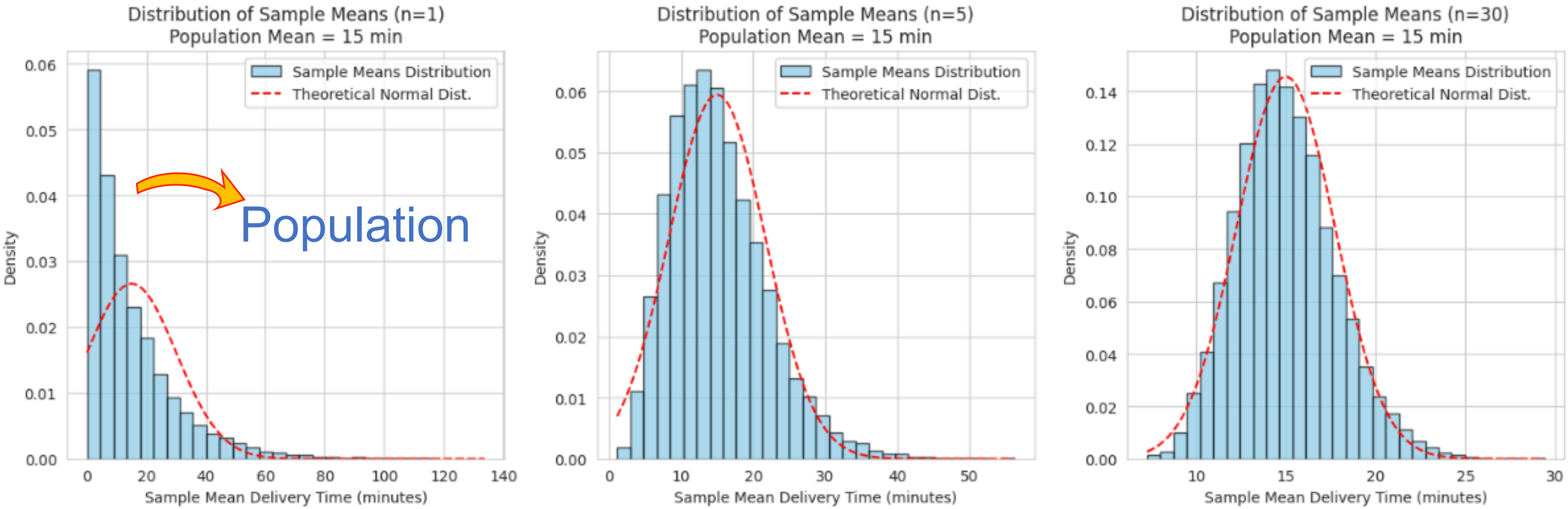

To observe the power of CLT, as an exercise: plugin a value for p in our polling example and apply the CLT.

Examples on Central Limit Theorem

Example 8.4: An electrical firm manufactures light bulbs that have a length of life that is approximately normally distributed, with mean equal to 800 hours and a standard deviation of 40 hours. Find the probability that a random sample of 16 bulbs will have an average life of less than 775 hours. Use norm.cdf(-2.5)=0.0062.

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$\frac{\bar{x} - \mu}{\sigma_{\bar{x}}} \rightarrow z$

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$$= 0.000233 \;//$$

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$$= P\left(Z < \frac{0.2 \times 10}{2}\right) = P(Z < 1) = 0.84$$